BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 4, Number 2, March 1981

CANONICAL MAPPINGS BETWEEN TEICHMÜLLER SPACES[1]

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Introduction. In an important survey article **[B10]** Bers reported on the state of knowledge of Teichmüller theory. There has been a lot of progress in the field since that time. The purpose of this paper is to summarize the recent work in one area of Teichmüller space theory. We will concentrate on the hyperbolic properties of Teichmüller spaces, and present as many consequences of this hyperbolicity as we can. The starting point of this study is Royden's **[Ro]** fundamental paper showing that the Teichmüller metric on $T(p, 0)$, $p \geqslant 2$, and the hyperbolic (Kobayashi **[Ko**, pp. 45–46]) metric are one and the same. The organization of the material of this paper follows that of an earlier joint paper with Clifford Earle **[EK1]**, except that an introductory section on history and motivation has been added.

We have neglected completely another area of Teichmüller space theory in which a tremendous amount of recent work has contributed greatly to our understanding of Riemann surfaces; namely, the study of fibrations and boundaries of Teichmüller spaces. I will only mention the people who have contributed to developments in this area: Abikoff, Bers, Earle, Hubbard, Jørgensen, Kerckhoff, Marden, Maskit, Masur, Thurston.

I am grateful to Bernard Maskit for reading an earlier version of this manuscript and for his many helpful suggestions regarding this article, in particular, and Kleinian groups, in general; and to Lipman Bers for his continual help, encouragement, and interest in all aspects of mathematics.

0. A short history and the most classical example.

0.1. The classical theory of moduli of Riemann surfaces originated with Riemann's observation that the conformal type of a compact Riemann surface of genus $p > 1$ depends on $3(p - 1)$ complex parameters, known as moduli. Yet, the fact that the *space of moduli of compact surfaces of genus* $p > 1$ forms a normal complex space was not proven until the early 1950's. The key step is passing to a cover of the space of moduli, known as the *Teichmüller space of genus* p, $T(p, 0)$. The space $T(p, 0)$ appears implicitly in the continuity arguments of Klein and Poincaré. It was constructed explicitly by Fricke **[FK]** and Fenchel-Nielsen **[FN]**. Fricke proved that $T(p, 0)$ is a

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This article is an expanded version of an Invited Address delivered at the 769th meeting of the American Mathematical Society held at Syracuse, New York on October 28, 1978; received by the editors July 15, 1980.

1980 *Mathematics Subject Classification*. Primary 30–02; Secondary 32G15, 32H20, 30C60, 30F10, 30F35.

[1]Research partially supported by NSF Grant MCS7801248.

0002-9904/81/0000-0102/$10.25

manifold real analytically equivalent to $\mathbf{R}^{6p-6}$. Teichmüller **[T1]**, **[T2]** introduced a natural metric on $T(p, 0)$. Ahlfors **[Ah3]**, **[Ah6]**, **[Ah7]** and Bers **[B1]**, **[B5]**, **[B6]** showed that $T(p, 0)$ has a natural complex structure, and Royden **[Ro]** completed this circle of ideas by recovering the natural metric from the natural complex structure.

A compact Riemann surface together with a standard system of generators for its fundamental group (called a *marked* surface) determines a point in $T(p, 0)$ and all points in the Teichmüller space arise this way. There is a natural group of biholomorphic mappings of $T(p, 0)$, the modular group, which identifies points in $T(p, 0)$ corresponding to conformally equivalent Riemann surfaces. The moduli space is $T(p, 0)$ factored by the modular group.

The model for $T(p, 0)$ is the upper half-plane which can be canonically identified with the Teichmüller space for surfaces of genus 1, $T(1, 0)$. We will outline briefly the theory of moduli of compact Riemann surfaces of genus 1 (elliptic function theory), showing the connections, similarities, and differences with the theory of moduli for surfaces of genus $p > 1$.

0.2. It is well known that every compact Riemann surface of genus 1 can be constructed as $X_\tau = \mathbf{C}/G_\tau$, where G_τ is the lattice generated by 1 and τ with $\tau \in U$, the upper half-plane. It is more convenient to view G_τ as a group of motions of $\mathbf{C}$, generated by the two translations $A\colon z \mapsto z + 1$ and B_τ: $z \mapsto z + \tau$. It should be remarked that this general uniformization theorem for tori does not depend on Koebe's uniformization theorem. To prove the uniformization theorem in genus 1, we need only use the more classical theorem of Abel (the map from a torus into its Jacobian variety is an isomorphism). The points in U form the Teichmüller space of genus 1: $T(1, 0) \cong U$.

0.3. In attempting to generalize the above construction to higher genus one immediately comes across two problems. The first problem concerns the *ad-hoc* nature of the above procedure (to be discussed in this section); and the second, the questions connected with uniformization of surfaces of higher genus (see §0.7).

Let us fix as a *base surface* the torus corresponding to $\tau = i = \sqrt{-1}$, $X_i = \mathbf{C}/G_i$. The group G_i can, of course, be identified with the fundamental group of X_i. Consider the space of isomorphisms of G_i onto other lattice subgroups of Aut $\mathbf{C}$ (that is, onto free discrete commutative subgroups of the group of automorphisms of $\mathbf{C}$, Aut $\mathbf{C}$). Two isomorphisms θ_1 and θ_2 will be called *equivalent* if they differ by an inner automorphism of Aut $\mathbf{C}$. It is easy to check that each such isomorphism is equivalent to a unique isomorphism θ with

$$\theta(A) = A, \quad \theta(B_i) = B_\tau, \quad \text{some } \tau \in U. \tag{0.3.1}$$

Thus we see that $T(1, 0)$ could as well have been defined more abstractly as a set of equivalence classes on the space of isomorphisms of G_i onto lattice subgroups of Aut $\mathbf{C}$. This concept can be used to define the Teichmüller spaces $T(p, 0)$ for $p \geqslant 2$ (see §1.12). However, in more general settings algebraic conditions alone will not suffice to construct deformation spaces.

0.4. Every isomorphism of G_i onto G_τ is induced by an orientation-preserving homeomorphism of **C** onto itself. In fact, the homeomorphism can be chosen to be quasiconformal and even affine (that is, of the form

$$z \mapsto az + b\bar{z} + c \tag{0.4.1}$$

with $a, b, c \in \mathbf{C}$, $a \neq 0$ and $|b/a| < 1$). The whole modern development of moduli of Riemann surfaces was initiated by Teichmüller with his study of extremal quasiconformal mappings **[T1]**, **[T2]**. (Quasiconformal maps will be discussed further in §1.2; extremal quasiconformal maps in §4.2.) Quasiconformal mappings already appeared in the earlier work of Grötzsch **[Gz]** and Ahlfors **[Ah1]**. They were introduced in a systematic way, before Teichmüller, by Morrey **[My]** and Lavrentieff **[La]**. Alternate (equivalent) definitions of quasiconformality were studied by Pfluger **[Pf]** and Ahlfors **[Ah2]**. (See Gehring's review **[Ge]** for more on the history of quasiconformality.) However, it was Teichmüller who first noticed the deep connection of quasiconformal mappings and function theory. The existence theorems for quasiconformal mappings with prescribed dilatation (Lavrentieff **[La]**, Bojarski **[Bo]**, Ahlfors-Bers **[AB]**) became, during the 1950's and 1960's, an essential tool in the study of variation of complex structures on (Riemann) surfaces. For a long time Teichmüller's work on extremal quasiconformal mappings was (despite its beauty and elegance)[2] unnecessary to study moduli (see Bers' survey article **[B10]**). However, in order to understand the hyperbolic nature of $T(p, 0)$, Teichmüller's characterization of extremal quasiconformal mappings is essential.

Using quasiconformal mappings, it is easy to define a metric on $T(p, 0)$. The affine mappings of (0.4.1) are extremal quasiconformal mappings, and one can define the Teichmüller distance on $T(1, 0)$ as follows.

Let X_1 and X_2 be two points in $T(1, 0)$. We represent X_j by a normalized (satisfying (0.3.1)) isomorphism $\theta_j\colon G_i \to \operatorname{Aut} \mathbf{C}$. Let f be an affine map of the form (0.4.1) that induces the isomorphism $\theta_2 \circ \theta_1^{-1}$. Define the Teichmüller distance τ on $T(1, 0)$ by

$$\tau(X_1, X_2) = \rho(0, b/a),$$

where ρ is the non-Euclidean (Poincaré) metric on the unit disk. This simple construction generalizes directly (see §4.2).

Quasiconformal mappings have important applications to the study of homeomorphisms between surfaces as well as to other branches of mathematics. In this connection see the very interesting papers by Bers **[B12]**, **[B13]**.

REMARK. See Earle **[E3]** for a proof of the existence of quasiconformal mappings with prescribed dilatation μ, for a restricted class of μ. Earle shows that for a special class, existence can be obtained as a consequence of the Banach space implicit function theorem (one of the few indispensable tools for analysis).

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[2]Two contributing factors might have been the fact that the proofs were not convincing (the ideas are all correct, nevertheless–Ahlfors **[Ah2]** and Bers **[B2]**), and the fact that the author of these beautiful theorems was far from beauty (and grace).

0.5. The Teichmüller space $T(p, 0)$, $p \geqslant 2$, is a generalization of the unit disk. The Teichmüller space is contractible (due to Fricke, Teichmüller), a domain of holomorphy (Bers-Ehrenpreis **[BE]**), and equivalent to a bounded domain (Bers **[B6]**). It, of course, satisfies a universal mapping property (Grothendieck **[Gk]**). It is tempting to try to translate as many properties of U to $T(p, 0)$. The upper half-plane carries a natural Riemannian metric of constant negative curvature–the Poincaré metric. The natural metric on $T(p, 0)$ is a Finsler metric. For a long time it was thought that the Teichmüller metric has negative curvature. However, the paper claiming to prove this result (Kravetz **[Kz]**) has a serious mistake, discovered by Linch **[Li]**. Recently, Masur **[Mr1]** has shown that this metric is not of negative curvature. Despite this, the Teichmüller metric tries to behave as if it were of negative curvature (see §3.5).

In many ways the boundaries (there are many!) of Teichmüller space behave like the boundary $\mathbf{R} \cup \{\infty\}$ of U (see for example, Bers **[B14]**, Abikoff **[Ab]**). We will, however, not pursue this line much further (see §0.10).

0.6. We have seen that the Teichmüller space $T(1, 0)$ with its canonical metric can be identified with the upper half-plane U with its canonical metric (the Poincaré metric). The result for arbitrary genus is remarkably similar (see §§4.2, 4.7). The space $T(1, 0)$ represents marked tori. To obtain the "space of tori" (= the "space of moduli"), one has to identify the points of $T(1, 0) \cong U$ that represent conformally equivalent tori. Using the identifications of §0.2, two points τ and $\tau' \in U$ represent the same torus if and only if we can find an affine map F that is complex analytic and conjugates G_τ onto $G_{\tau'}$. It is a simple exercise to show that such an F exists if and only if

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad \text{with } a, b, c, d \in \mathbf{Z}, \text{ and } ad - bc = 1.$$

Thus the *modular group* $\Gamma = SL(2, \mathbf{Z})/\{\pm I\}$ acts naturally on $T(1, 0)$ to produce the space of moduli for surfaces of genus 1. The fact that U/Γ is a manifold ($\cong \mathbf{C}$ with two distinguished points or $\mathbf{C} \cup \{\infty\}$ with three distinguished points (of which one is not there)) is an accident of dimension 1. However, in general, a discrete group acts analytically on $T(p, 0)$ to produce the space of moduli for surfaces of genus $p \geqslant 1$. In genus one (bigger than one) this group can (must) be defined more abstractly in terms of certain classes of automorphisms of the base group (see §3).

The automorphism group of $T(1, 0)$ is, of course, $SL(2, \mathbf{R})/\{\pm I\}$, a real Lie group. For higher genus, however, the modular group is (essentially) the full group of automorphisms of $T(p, 0)$, $p \geqslant 2$ (see §5.6). The Teichmüller spaces thus provide examples of contractible domains of holomorphy with discrete automorphism groups.

0.7. The complex structure of $T(1, 0)$ is quite natural because every torus is uniformized by a subgroup of the (three-dimensional) complex Lie group Aut $\mathbf{C} \cong$ (2×2)-upper triangular matrices of the form $\binom{a\ b}{0\ 1}$ with $a \in \mathbf{C}^*$, $b \in \mathbf{C}$. However, a surface of genus $p \geqslant 2$ is uniformized by a Fuchsian group–a subgroup of $SL(2, \mathbf{R})/\{\pm I\}$. Since $SL(2, \mathbf{R})$ is *only* a real Lie group, it is not at all obvious how to give a complex structure to a space of isomorphisms from a fixed Fuchsian group Γ onto other Fuchsian groups. Ahlfors gave $T(p, 0)$ a complex structure by constructing specific local coordinates using periods of abelian differentials **[Ah3]**.

A very simple and beautiful idea of Bers **[B4]** provides an alternate construction. A Fuchsian group Γ represents two Riemann surfaces U/Γ and its mirror image (U^*/Γ, where U^* is the lower half-plane). If one varies the Fuchsian group, each of the surfaces changes–but the change of one completely controls what happens to the other surface. These changes in complex structure of the underlying Riemann surfaces are *only* real analytic.

Bers **[B4]** observed that for any two surfaces X_1 and X_2 of the same genus $p > 1$, there is a group that algebraically looks just like a Fuchsian group and represents precisely the two surfaces X_1 and X_2. These groups (called *quasi-Fuchsian*) act on two (topological) discs in $\mathbf{C} \cup \{\infty\}$. Bers **[B5]** constructs the Teichmüller space $T(p, 0)$ by keeping fixed the complex structure of the surface produced by the quasi-Fuchsian group acting on one of these two disks, and varying the complex structure of the surface produced by the action of the group on the other disk. This idea has very wide applicability (see §1.2) and is the basis of much of the work on deformation spaces of Kleinian groups (and almost all of this paper).

0.8. The Teichmüller space $T(1, 0) \cong U$ represents the space of marked tori. To get the surfaces into the picture, we construct various fiber spaces. We view $U \times \mathbf{C} \xrightarrow{\pi_1} U$ (π_1 = projection onto first coordinate) as a fiber space over $T(1, 0)$ with fiber over τ ($\pi_1^{-1}(\tau) = \mathbf{C}$) the holomorphic universal covering space of the torus $X_\tau = \mathbf{C}/G_\tau$. We let $\mathbf{Z}^2$ act on $U \times \mathbf{C}$ by

$$(n, m)(\tau, z) = (\tau, z + n + m\tau), \qquad n, m \in \mathbf{Z}, \tau \in U, z \in \mathbf{C}.$$

The quotient space $V(1, 0) = (U \times \mathbf{C})/\mathbf{Z}^2$ is a fiber space over U with projection

$$\pi \colon V(1, 0) \to T(1, 0)$$

induced by π_1. Note that $\pi^{-1}(\tau) = X_\tau$.

(As an exercise, let $\Gamma = (SL(2, \mathbf{Z})/\{\pm I\}) \times \mathbf{Z}^2$ act on $U \times \mathbf{C}$ by

$$(g, n, m)(\tau, z) = (g\tau, z + n + mg\tau),$$
$$g \in SL(2, \mathbf{Z})/\{\pm I\}, \qquad n, m \in \mathbf{Z}, \tau \in U, z \in \mathbf{C},$$

and compute $(U \times \mathbf{C})/\Gamma$.)

The holomorphic sections s of the map π (that is, holomorphic maps s: $T(1, 0) \to V(1, 0)$ such that $\pi \circ s = \mathrm{id}$) are in canonical correspondence with the holomorphic maps $f\colon U \to \mathbf{C}$ (given f, set $s(\tau) = (\tau, f(\tau))$), and hence not much can be said about sections of π. Note that a section of π is a choice of a point on a torus, with the choice depending holomorphically on moduli. The abundance of sections of π is, of course, not surprising since every torus is homogeneous (it has a transitive group of conformal automorphisms). It is precisely this homogeneity that allows us to identify $T(1, 0)$, the Teichmüller space of tori, with $T(1, 1)$, the Teichmüller space of once-punctured tori. In this setting

$$V(1, 1)' = (U \times \mathbf{C})'/\mathbf{Z}^2,$$

where

$$(U \times \mathbf{C})' = \{(\tau, z) \in U \times \mathbf{C}; z \neq n + m\tau, \text{all } n, m \in \mathbf{Z}\}.$$

(The reason for the ′ will become apparent in §2.3.) The holomorphic sections of π: V(1, 1)′ → T(1, 1) (are already interesting and) are in canonical correspondence with holomorphic functions f: $U \to \mathbf{C}$ such that

$$f(\tau) \neq \tau + n + m\tau, \text{all } \tau \in U, \text{all } n, m \in \mathbf{Z}. \tag{0.8.1}$$

Pick any $z_0 \in \mathbf{C} \setminus \{n + m\tau: n, m \in \mathbf{Z}\}$. Then by properly choosing $(a, b) \in \mathbf{R}^2 \setminus \mathbf{Z}^2$, $f(\tau) = a + b\tau$ will satisfy (0.8.1). Are there any other holomorphic functions that satisfy (0.8.1)? The answer is no (Earle **[E1]**, for a proof see Earle-Kra **[EK1]**). Despite the classical (elliptic function-theoretic) presentation of the problem, the only proof (in print) relies heavily on Teichmüller space theory (its hyperbolic properties). The question of choosing points (holomorphically) on compact surfaces of genus $p \geqslant 2$ has been completely solved (most of the times it cannot be done). See §6.

0.9. We have marked a compact Riemann surface X by choosing a set of generators for its fundamental group. We are led this way to Teichmüller space. We can alternately mark a surface by choosing a set of generators for the first homology group and obtain *Torelli spaces*. The Torelli space is always a factor space of the Teichmüller space; these two spaces coincide only in genus one (because the fundamental group of a torus is commutative). We will discuss (very briefly) the use of homology in moduli problems in §7.

0.10. The Teichmüller spaces are generalizations of U, the upper half-plane, and the modular groups are generalizations of Fuchsian groups. The recent works of Thurston, Masur, Hubbard explore this connection. The modular group acts on certain boundary points of Teichmüller space (Bers **[B14]**, Abikoff **[Ab]**, Earle-Marden **[EM]**). This action is currently under active investigation and strengthens the model for Teichmüller spaces that we have been describing.

At this point we will stop the general discussion and proceed to describe more precisely the objects of interest.

1. Deformation spaces of Kleinian groups.

1.1. Let G be a group of Mobius transformations ($G \subset \mathbf{P}SL(2, \mathbf{C}) =$ Mob); that is, a group of self-mappings

$$z \mapsto \frac{az + b}{cz + d} \quad \text{with } a, b, c, d \in \mathbf{C}, ad - bc = 1,$$

of the extended complex planes, $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. The group G acts *discontinuously* at $z \in \hat{\mathbf{C}}$ if the stability subgroup of G at z,

$$G_z = \{g \in G; gz = z\}$$

is finite, and there exists a neighborhood D of z such that

$$g(D) = D \quad \text{all } g \in G_z \quad \text{and} \quad g(D) \cap D = \varnothing \quad \text{all } g \in G \setminus G_z.$$

The *set of discontinuity*, $\Omega = \Omega(G)$, of G consists of those $z \in \hat{\mathbf{C}}$ such that G acts discontinuously at z. It is an open G-invariant subset of $\hat{\mathbf{C}}$. Its complement, $\Lambda = \Lambda(G)$, is called the *limit set* of G. The group G is called *Kleinian* provided $\Omega \neq \varnothing$, and nonelementary provided card $\Lambda > 2$. If G is nonelementary, the Λ is a perfect, closed subset of $\hat{\mathbf{C}}$. If G is Kleinian then Ω, in general, consists of infinitely many components, and Ω/G is a (countable, in general) union of Riemann surfaces.

The components of Ω are also called *components* of G. Not much is known about the structure of Ω/G, unless one makes some further assumptions about the structure of G. A fundamental result of Ahlfors [**Ah8**], known as Ahlfors' finiteness theorem, asserts that if G is a finitely-generated Kleinian group, then Ω/G is a finite union of Riemann surfaces that are compact, except for finitely many missing points, and the canonical projection $\Omega \to \Omega/G$ is ramified over finitely many points (the ramification points $z \in \Omega$ corresponds to the points with nontrivial stabilizers G_z).

Ahlfors' deep and beautiful result was proved in the sixties, and it is central and basic for all subsequent work on Kleinian groups. The proof of the finiteness theorem involves the use of (group) cohomology and depends on a delicate smoothing operator.

In general, a component Δ of Ω is left fixed (as a set) by a subgroup of G. If $g\Delta = \Delta$ for all $g \in G$, then Δ is called an *invariant component* of G, and G is called a *function* group. The most important function groups are the *Fuchsian groups*; that is, those function groups where Δ is a disk (or half-plane).

Throughout the rest of this paper G will denote a finitely-generated nonelementary Kleinian group with an invariant component Δ. In this case the limit set Λ of G is precisely the boundary of Δ. (For more on the elementary properties of Kleinian groups, the reader is referred to Ford's [**Fo**] or Lehner's [**Le**] book. A proof of Ahlfors' finiteness theorem may be found in the author's book [**Kr1**].)

1.2. Let $L^\infty(G)$ be the space of *Beltrami differentials* for G with support in Δ; that is, the closed subspace of $L^\infty(\Delta, \mathbf{C})$ consisting of all $\mu \in L^\infty(\Delta, \mathbf{C})$ with

$$(\mu \circ g)\frac{\overline{g'}}{g'} = \mu \quad \text{all } g \in G.$$

We consider each $\mu \in L^\infty(G)$ to be defined on all of $\hat{\mathbf{C}}$ with $\mu|(\hat{\mathbf{C}} \setminus \Delta) = 0$. (Every Beltrami differential μ (with arbitrary support in $\mathbf{C} \cup \{\infty\}$) for an arbitrary finitely generated Kleinian group, must vanish on Λ by a recent result of Sullivan [**Su**].) The set of *Beltrami coefficients* $M(G)$ for G with support in Δ is an open unit ball in $L^\infty(G)$.

A homeomorphism ω of $\hat{\mathbf{C}}$ onto itself is *normalized* if it fixes 0, 1, ∞, and is *μ-conformal* if it satisfies the Beltrami equation $\omega_{\bar{z}} = \mu\omega_z$. For each $\mu \in M(G)$, there is a unique normalized μ-conformal automorphism w^μ of $\hat{\mathbf{C}}$ (see Ahlfors-Bers [**AB**]). Note that each w^μ is conformal on the interior of $\hat{\mathbf{C}} \setminus \Delta$.

A quasiconformal homeomorphism ω is *compatible* with G (modulo Δ) if ω is conformal on the complement of Δ and

$$\omega \circ g \circ \omega^{-1} \in \text{Mob} \quad \text{all } g \in G.$$

A quasiconformal ω is compatible with G if and only if $\omega = \alpha \circ w^\mu$ for some $\alpha \in \text{Mob}$, some $\mu \in M(G)$. A compatible ω induces an isomorphism θ_ω of G

onto the Kleinian group $G^\omega = \omega G \omega^{-1}$, with invariant component $\omega(\Delta)$, defined by

$$\theta_\omega(g) = \omega \circ g \circ \omega^{-1}, \qquad g \in G.$$

We say that two compatible quasiconformal maps ω and ω_1 are *equivalent* if there exists an $\alpha \in$ Mob such that $\theta_{\omega_1} = \theta_{\alpha \circ \omega}$.

The *deformation space* $\hat{T}(G)$ is the set of equivalence classes of compatible quasiconformal maps. We note that the equivalence class of a quasiconformal compatible ω, depends only on the Beltrami coefficient μ of ω. Thus, we have a well-defined surjective mapping Φ: $M(G) \to \hat{T}(G)$. We give $\hat{T}(G)$ a topology and complex structure by declaring Φ to be continuous and holomorphic. Every deformation space $\hat{T}(G)$ contains a distinguished point $\Phi(0)$. We call this point the *origin* or *zero* point of $\hat{T}(G)$ and denote it by 0.

It is quite easy to see that for $\mu, \nu \in M(G)$, $\Phi(\mu) = \Phi(\nu)$ if and only if $w^\mu|\Lambda = w^\nu|\Lambda$, provided $\{0, 1, \infty\} \subset \Lambda$.[3] We shall introduce the following abbreviations

$$G^\mu = G^{w^\mu}, \qquad \Delta^\mu = w^\mu(\Delta), \qquad \theta_\mu = \theta_{w^\mu}.$$

From now on, we assume that G is normalized so that 0, 1, ∞ are limit points of G.

(The deformation spaces $\hat{T}(G)$ can be defined for arbitrary Kleinian groups–not just function groups. See, for example, Kra **[Kr2]**. The deformation space $\hat{T}(G)$ defined above corresponds to the space $\hat{T}(G, \Delta)$ of **[Kr2]**.)

REMARK. The space of Beltrami differentials $L^\infty(G)$, for the group G with support in Δ, projects to a space of Beltrami differentials $L^\infty(\Delta/G)$ on the quotient surface Δ/G. These projected differentials are bounded $(-1, 1)$-forms (that is, $\mu \in L^\infty(\Delta/G)$ is an assignment of a bounded function μ to each local coordinate z so that $\mu(z)d\bar{z}/dz$ is invariantly defined).

Similarly (because the norm is independent of local coordinates) $M(G) \cong M(\Delta/G)$.

1.3. We shall be concerned mostly with two special cases. The most important (classical) case is when G is a finitely-generated Fuchsian group *of the first kind* operating on the upper half-plane U. Such groups will always be denoted by the letter Γ; and we will write $T(\Gamma)$ for $\hat{T}(\Gamma)$; and call $T(\Gamma)$, the *Teichmüller space of* Γ. (There is a more general class of Fuchsian groups–called of the second kind.) If G is a discrete subgroup of Mob and G leaves a disk U invariant, then G is (classically) called a Fuchsian group. Such a group always acts discontinuously on U. However, U need not be a component of Ω. According to our definition in §1.1, such a group would not be Fuchsian. However, classically it is called a Fuchsian group of the *second kind*, whenever Λ is a proper subset of the boundary of U. For nonelementary Fuchsian groups of the second kind (as a matter of fact, as long as card $\Lambda(G) > 1$), Ω is connected but not simply connected).

The importance of these Fuchsian groups is due to the fact that they uniformize almost all Riemann surfaces (see §1.7).

[3]In **[Kr2]**, we assumed throughout that 0, 1, $\infty \in \Lambda$. However, we omitted to mention this normalization.

If Γ is a finitely-generated Fuchsian group of the first kind, if $\mu \in M(\Gamma)$, and if $A \in$ Mob, then $\omega = A \circ w^{\mu}$ conjugates Γ onto $\Gamma^{\omega} = \omega\Gamma\omega^{-1}$; the group Γ^{ω} is called *quasi-Fuchsian*. Maskit has shown that every finitely-generated Kleinian group with two invariant components (no finitely-generated Kleinian group can have more than two invariant components) is quasi-Fuchsian (see **[KM1]**). Thus for a non-quasi-Fuchsian function group G, the invariant component Δ is unique. For quasi-Fuchsian groups, there are two choices. For the theory to be developed here, the choice of components is not material.

The second of our special cases lies at the other end of the spectrum of uniformizations of Riemann surfaces. Let D be a region in $\hat{\mathbf{C}}$ bounded by $2p$ (with $p \geqslant 2$) disjoint simple closed curves $C_1, C_1', \ldots, C_p, C_p'$. For $j = 1, \ldots, p$, let A_j be a Mobius transformation with $A_j(D) \cap D = \varnothing$ and $A_j(C_j) = C_j'$ (such Mobius transformations always exist if C_j and C_j' are circles or straight lines). Let G be the group generated by $A_1, \ldots, A_p$. The group G is called a *Schottky* group. It is a free group on p generators, and if we let

$$\Delta = \bigcup_{g \in G} g(D'),$$

where $D' = D \cup C_1 \cup \cdots \cup C_p$, then Δ is the entire region of discontinuity of G and Δ/G represents a closed surface of genus $p \geqslant 2$.

Conversely, every finitely-generated, purely loxodromic, free Kleinian group is a Schottky group by a theorem of Maskit **[Ms1]**.

1.4. Let $Q(G)$ denote the Banach space of holomorphic functions φ on Δ with

$$(\varphi \circ g)(g')^2 = \varphi \quad \text{all } g \in G, \tag{1.4.1}$$

and norm

$$\|\varphi\| = \frac{1}{2}\int\int_{\Delta/G} |\varphi(z)\, dz \wedge d\bar{z}| < \infty.$$

Define a natural pairing

$$(\varphi, \mu)_G = \frac{1}{2}\int\int_{\Delta/G} \varphi(z)\mu(z)|dz \wedge d\bar{z}|, \qquad \varphi \in Q(G), \mu \in L^{\infty}(G).$$

This pairing establishes a canonical isomorphism between the dual space $Q(G)^*$ of $Q(G)$ and $L^{\infty}(G)/Q(G)^{\perp}$, where

$$Q(G)^{\perp} = \{\mu \in L^{\infty}(G);\ (\varphi, \mu)_G = 0, \text{all } \varphi \in Q(G)\}.$$

THEOREM. *The deformation space $\hat{T}(G)$ has a unique complex analytic manifold structure so that the map Φ: $M(G) \to \hat{T}(G)$ is holomorphic with local holomorphic sections. The null space of the differential $\Phi'(\mu)$ at $\mu \in M(G)$ is $Q(G^{\mu})^{\perp}$.*

The fact that for Fuchsian groups Γ, $T(\Gamma)$ is a complex manifold is deep and fundamental work of Ahlfors and Bers (see Bers **[B6]**). The extension to the more general situation is more routine (see Bers [**B8**], Maskit [**Ms3**], and Kra [**Kr2**]).

Since $Q(G^\mu)$ is finite dimensional, the previous theorem has a

COROLLARY. *The tangent space to* $\hat{T}(G)$ *at* $\Phi(\mu)$ *is* (*canonically isomorphic to*) $Q(G^\mu)^*$. *The cotangent space is* $Q(G^\mu)$.

1.5. By $B(G)$ we will denote the Banach space of *bounded quadratic differentials* on the interior Δ^* of $\hat{\mathbf{C}} \setminus \Delta$ for the group G; that is, the set of holomorphic functions φ on Δ^* satisfying (1.4.1) with norm

$$\|\varphi\|' = \sup\{\lambda(z)^{-2}|\varphi(z)|;\ z \in \Delta^*\} < \infty,$$

where $\lambda(z)|dz|$ is the Poincaré metric on Δ^* (each component of Δ^* is conformally equivalent to U and, hence, carries a Poincaré metric) of constant negative curvature -1.

WARNING. The open set Δ^* may be empty. It is empty for Schottky groups and degenerate groups, for example. A finitely-generated nonelementary Kleinian group is called *degenerate* if Ω, its region of discontinuity, is connected and simply connected. A lot of the mystery about Kleinian groups is due to the presence of these objects. Degenerate groups first appear in the companion paper of Bers [**B9**] and Maskit [**Ms2**].

For $\mu \in M(G)$, let φ^μ be the Schwarzian derivative of $w^\mu|\Delta^*$. Then $\mu \mapsto \varphi^\mu$ defines a holomorphic mapping of $\hat{T}(G)$ into $B(G)$. This mapping is always locally surjective (as a consequence of Bers [**B6**]). It is injective if and only if $\dim B(G) = \dim Q(G)$ if and only if G is a quasi-Fuchsian group (Maskit [**Ms4**]). For quasi-Fuchsian groups, the mapping is the *Bers embedding* of $\hat{T}(G)$ in $B(G)$. The image is always a bounded domain.

REMARK. As remarked in §1.4, the fact that $T(\Gamma)$, for a Fuchsian group Γ, is a complex manifold is a deep fact. However for Γ finitely generated of the first kind, the manifold structure of $T(\Gamma)$ and its embedding as an *open* subset of $B(\Gamma)$ follows rather easily from the fact that homeomorphic solutions of the Beltrami equation depend holomorphically on parameters. We outline below two such proofs.

FIRST PROOF. 1. The fact that for fixed $z \in \mathbf{C}$, $M(G) \ni \mu \mapsto w^\mu(z) \in \mathbf{C}$ is a holomorphic mapping [**AB**], shows that the Bers embedding is holomorphic [**B6**]. It is easily seen to be one-to-one when G is quasi-Fuchsian [**B6**].

2. A calculation (see [**B6**]) shows that the derivative of the mapping Φ at the origin is the linear operator L defined by

$$(L\mu)(z) = \frac{6}{2\pi i}\int\int_\Delta \frac{\mu(\zeta)d\zeta \wedge d\bar{\zeta}}{(\zeta - z)^4}, \qquad \mu \in L^\infty(G), z \in \Delta^*.$$

3. If we let λ_1 be the Poincaré metric on Δ, then for quasi-Fuchsian groups G,

$$(\mathfrak{L}\varphi)(z) = \frac{6}{2\pi i}\int\int_\Delta \frac{\lambda_1(\zeta)^{-2}\overline{\varphi(\zeta)}\, d\zeta \wedge d\bar{\zeta}}{(\zeta - z)^4}, \qquad \varphi \in Q(G), z \in \Delta^*,$$

is a surjective linear isomorphism of $Q(G)$ onto $B(G)$. After computing dimensions, it suffices to show only injectivity of $\mathfrak{L}$. This is quite easy, **[B6]** or **[B7]**. The fact that $\mathfrak{L}$ is surjective for the trivial group is a deep result **[B6]**.

4. The implicit function theorem now shows that Φ maps $T(\Gamma)$ biholomorphically onto an open subset of $B(\Gamma)$ for finitely generated Fuchsian Γ of the first kind, and that the mapping Φ: $M(\Gamma) \to B(\Gamma)$ has local holomorphic sections.

SECOND PROOF. Alternately, Ahlfors-Weill **[AW]** show by explicit construction that Φ: $M(\Gamma) \to B(\Gamma)$ covers a neighborhood of the origin. This proves that origin is a (finite dimensional) manifold point of $T(\Gamma)$. Allowable mappings (see §3.1) then show that every point is a manifold point. Invariance of domain can now be used to show that the image of $T(\Gamma)$ in $B(\Gamma)$ is an open set.

1.6. Let h: $U \to \Delta$ be a holomorphic universal covering map, and define the *Fuchsian model* Γ for the action of G on Δ:

$$\Gamma = \{\gamma \in (\text{holomorphic self-maps of } U);\ h \circ \gamma = g \circ h \text{ for some } g \in G\}.$$

The above definition leads to an exact sequence of groups and group homomorphisms

$$\{1\} \to H \hookrightarrow \Gamma \xrightarrow{\theta} G \to \{1\},$$

where θ is defined by

$$h \circ \gamma = \theta(\gamma) \circ h \quad \text{all } \gamma \in \Gamma,$$

and where

$$H = \text{kernel of } \theta \cong (\text{fundamental group of } \Delta).$$

Furthermore, $\Delta \cong U/H$, and the canonical projections make the following diagram commute.

$$\begin{array}{ccc} U & \xrightarrow{h} & \Delta \\ \downarrow & & \downarrow \\ U/\Gamma & \xrightarrow{\cong} & \Delta/G \end{array}.$$

In particular, U/Γ and Δ/G are equivalent as Riemann surfaces with ramification points.

Now, the space of Beltrami differentials $L^\infty(G)$ is canonically isomorphic to $L^\infty(\Delta/G) \cong L^\infty(U/\Gamma)$. Thus, it is natural to identify $L^\infty(G)$ with $L^\infty(\Gamma)$. The identification is given by the surjective linear isometry

$$h^*: L^\infty(\Gamma) \to L^\infty(G)$$

defined by

$$(h^*\mu) \circ h = \mu \frac{\overline{h'}}{h'}, \qquad \mu \in L^\infty(\Gamma).$$

The mapping h^* allows one to construct a mapping between deformation spaces $T(\Gamma) \to \hat{T}(G)$.

THEOREM. *The Teichmüller space $T(\Gamma)$ is a holomorphic universal covering space of the deformation space $\hat{T}(G)$.*

For more details and generalizations of the above see the articles in the 1974 *Crash course on Kleinian groups* **[BK]**.

1.7. Let U_Γ be the set of points in the upper half-plane U which are not fixed by any elliptic element of the Fuchsian group Γ (assumed to be finitely generated of the first kind). Then U_Γ/Γ is a compact Riemann surface of genus p with n punctures. The pair (p, n) is called the *type* of Γ or of U_Γ/Γ. A group of type (p, n) exists if and only if U_Γ/Γ has negative Euler characteristic; that is, if and only if

$$2p - 2 + n > 0. \tag{1.7.1}$$

A theorem of Bers-Greenberg **[BG]** (see also Marden **[Md1]** and Earle-Kra **[EK1]**) asserts that $T(\Gamma)$ and $T(\Gamma')$ are biholomorphically equivalent whenever Γ and Γ' have the same type. We therefore define the Teichmüller space $T(p, n)$ as $T(\Gamma)$ for some group Γ of type (p, n). We consider only those types (p, n) that satisfy (1.7.1).

REMARKS. (1) The "excluded" types can be handled by easier and classical methods. It is quite easy to prove that one can define Teichmüller spaces for these excluded cases that yield

$$T(1, 0) \cong T(1, 1), \qquad T(0, 2) = T(0, 1) = T(0, 0) = \text{a point}.$$

(2) If Γ is of type (p, n), then the elements of $Q(\Gamma)$ project to integrable holomorphic quadratic differentials on U_Γ/Γ; that is, assignments of holomorphic functions φ to local coordinates z so that $\varphi(z)dz^2$ is invariantly defined on U_Γ/Γ. These differentials may have simple poles at the punctures of U_Γ/Γ. By the Riemann-Roch theorem

$$\dim_{\mathbf{C}} Q(\Gamma) = 3p - 3 + n.$$

(3) For arbitrary G, the mapping h introduced in §1.6 induces an isometric isomorphism $h_*\colon Q(G) \to Q(\Gamma)$ defined by

$$h_*\varphi = (\varphi \circ h)(h')^2, \qquad \varphi \in Q(G).$$

Thus we also have computed $\dim_{\mathbf{C}} Q(G)$.

(4) Fuchsian groups Γ of type (p, n) have a finer numerical invariant. For each puncture x_j on U_Γ/Γ, let $\nu_j = \infty$ if $x_j \notin U/\Gamma$, and let $\nu_j =$ order of stability subgroup of a preimage of x_j in U, otherwise. By reordering the n punctures we may assume

$$2 \leqslant \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_n \leqslant \infty.$$

The *signature* of Γ is the collection

$$(p; \nu_1, \ldots, \nu_n). \tag{1.7.2}$$

Further, two Fuchsian groups (finitely generated of the first kind) are quasiconformally equivalent if and only if they have the same signature.

1.8. Let X be a Riemann surface of type (p, n). The Teichmüller space $T(p, n)$ can be defined completely in terms of the Riemann surface X. Let f_j: $X \to X_j$ be a quasiconformal homeomorphism of X onto another Riemann surface X_j (necessarily of the same type) for $j = 1, 2$. We say that f_1 is *equivalent* to f_2 if there exists a conformal mapping $h\colon X_1 \to X_2$ such that

$f_2^{-1} \circ h \circ f_1$ is homotopic to the identity self-mapping of X. It is not hard to see that this set of equivalence classes yields a Teichmüller space $T(X)$. The topology and complex structure of $T(X)$ can be defined directly. However, the easiest way to prove that $T(X)$ is a complex manifold is to use the identification $T(U_\Gamma/\Gamma) = T(\Gamma)$; valid for arbitrary (finitely-generated) Fuchsian groups (of the first kind).

REMARK. Since we are dealing with low (real)-dimensional manifolds (two), the concepts of homotopy and isotopy agree (see Birman **[Bi]** and the papers quoted there). Thus we could, alternately, require the mapping $f_2^{-1} \circ h \circ f_1$ introduced previously, to be isotopic to the identity.

1.9. It is rather easy to verify that, for Schottky groups G and G', the deformation spaces $\hat{T}(G)$ and $\hat{T}(G')$ are biholomorphically equivalent if and only if Δ/G and Δ'/G' have the same genus (the genus of Δ/G is the number of free generators of G). Thus, we define the *Schottky space* of genus p, $S(p)$, to be the deformation space $\hat{T}(G)$ for some Schottky group G of genus $p \geqslant 2$.

1.10. Let G be generated by N-elements, $g_1, \ldots, g_N$. Any isomorphism θ: $G \to \mathbf{P}SL(2, \mathbf{C})$ can be viewed as a point

$$(\theta(g_1), \ldots, \theta(g_n)) \in \mathbf{P}SL(2, \mathbf{C})^N.$$

The group $\mathbf{P}SL(2, \mathbf{C})$ acts on $\mathbf{P}SL(2, \mathbf{C})^N$ by conjugation, and the quotient is a complex space. A point in $\hat{T}(G)$ is an equivalence class (modulo conjugation) of an isomorphism θ of G into $\mathbf{P}SL(2, \mathbf{C})$ that satisfies some additional (geometric) conditions. Thus we have a well-defined mapping

$$\hat{T}(G) \to \mathbf{P}SL(2, \mathbf{C})^N/\mathbf{P}SL(2, \mathbf{C}).$$

This mapping is holomorphic and injective.

1.11. Every point of $T(p, 0)$, $p \geqslant 2$, represents a compact Riemann surface X of genus p together with a "marking". This marking is a choice of generators for $\pi_1(X)$, the fundamental group of X; hence a choice of generators for $H_1(X)$, the first homology group for X (with integral coefficients). Thus for each point in $T(p, 0)$ we can construct a canonical basis for the holomorphic differentials of the first kind, and also a period matrix. Hence we have a mapping of $T(p, 0)$ into period matrices (points in the Siegal upper half-space of genus p). This mapping is holomorphic (see Rauch **[Ra]**). (See also §7 for more details.)

1.12. The deformation spaces $\hat{T}(G)$ have been defined in terms of certain isomorphisms of G onto other Kleinian groups. An isomorphism θ: $G \to G'$ between Kleinian groups is called *geometric*[4] if there exists a quasiconformal automorphism f of $\hat{\mathbf{C}}$ such that

$$\theta(g) = f \circ g \circ f^{-1} \quad \text{all } g \in G.$$

It is of great interest to determine necessary and sufficient conditions for an isomorphism to be geometric. This question has been investigated by Maskit **[Ms5]** and Marden **[Md2]**, **[Md3]**.

[4]The only difference between a geometric isomorphism and one induced by a compatible quasiconformal map is that, in the first case, the quasiconformal map is *not* required to be conformal on the interior of $\hat{\mathbf{C}} \setminus \Delta$.

2. Forgetful maps.

2.1. There is a canonical map f_k: $T(p, n + k) \to T(p, n)$ for every integer $k \geqslant 1$ which corresponds heuristically to "forgetting k punctures". We will study this map in some detail.

Let X be a Riemann surface of type (p, n), and let $x_1, \ldots, x_k$ be k distinct points on X. Let $X' = X \setminus \{x_1, \ldots, x_k\}$. Choose holomorphic universal covering maps t: $U \to X$ and t': $U \to X'$ with cover groups Γ and Γ' of types (p, n) and $(p, n + k)$, respectively. The inclusion map j: $X' \to X$ lifts to a map h: $U \to U$ which makes the following diagram commute.

$$\begin{array}{ccc} U & \xrightarrow{\;h\;} & U \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t} \\ X' & \xrightarrow{\;j\;} & X \end{array}$$

Further, h is a universal cover of $U' = t^{-1}(X')$ and induces a surjective homomorphism θ: $\Gamma' \to \Gamma$ defined by

$$h \circ \gamma = \theta(\gamma) \circ . \quad \text{all } \gamma \in \Gamma'.$$

Using h we define norm-preserving isomorphisms

$$h_*: Q(\Gamma) \to Q(\Gamma'), \qquad h^*: L^\infty(\Gamma') \to L^\infty(\Gamma)$$

by the formulae

$$h_*\varphi = (\varphi \circ h)(h')^2, \qquad (h^*\mu) \circ h = \mu h'/\overline{h'}, \qquad \varphi \in Q(\Gamma), \mu \in L^\infty(\Gamma').$$

These maps are adjoints in the sense that

$$(\varphi, h^*\mu)_\Gamma = \left(h_*\varphi, \mu\right)_{\Gamma'}, \qquad \varphi \in Q(\Gamma), \mu \in L^\infty(\Gamma').$$

As before, h^* projects to a well-defined holomorphic mapping

$$f_k: T(\Gamma') \to T(\Gamma). \tag{2.1.1}$$

Let $\mu \in M(\Gamma')$ and set $\nu = h^*\mu$. By its construction f_k carries $\Phi(\mu)$ in $T(\Gamma')$ to $\Phi(\nu)$ in $T(\Gamma)$. Further, there is now a unique holomorphic mapping $\tilde{h}$ which makes the following diagram commute.

$$\begin{array}{ccc} U & \xrightarrow{\;h\;} & U \\ {\scriptstyle w^\mu}\downarrow & & \downarrow{\scriptstyle w^\nu} \\ U^\mu & \xrightarrow{\;\tilde{h}\;} & U^\nu \end{array}$$

As before, we have well-defined mappings $\tilde{h}_*$ and $\tilde{h}^*$. The differential $f_k'(\Phi(\mu))$ is

$$\tilde{h}^*: L^\infty(\Gamma'^\mu)/Q(\Gamma'^\mu)^\perp \to L^\infty(\Gamma^\nu)/Q(\Gamma^\nu)^\perp.$$

THEOREM. *Let f_k: $T(\Gamma') \to T(\Gamma)$ be the forgetful map. The induced map of cotangent spaces is the inclusion map $\tilde{h}_*$ of $Q(\Gamma^\nu)$ in $Q(\Gamma'^\mu)$.*

REMARK. The forgetful map f_k has a very simple interpretation on the Riemann surface level. It is a map f: $T(X') \to T(X)$. A point θ of $T(X')$ is represented by an equivalence class of quasiconformal homeomorphisms $\omega: X' \to X_1'$. Since every quasiconformal map is extendible across punctures, ω can be viewed as a quasiconformal mapping of X onto another surface. The equivalence class of this extended mapping is $f(\theta)$.

The induced mapping on cotangent vectors at the origin is indeed the inclusion of $Q(X)$ in $Q(X')$. A similar interpretation is valid at an arbitrary point of $T(X')$.

2.2. For each $\mu \in M(G)$, the domain $w^{\mu}(\Delta)$ depends only on $\Phi(\mu)$. We can therefore define the *Bers fiber space*

$$F(G) = \{(\Phi(\mu), z) \in \hat{T}(G) \times \mathbf{C};\ \mu \in M(G), z \in \Delta^{\mu}\}.$$

THEOREM (BERS [**B11**]). *Let Γ be a Fuchsian group of type (p, n) with no elliptic elements. There is a biholomorphic map of $T(p, n+1)$ onto $F(\Gamma)$ which identifies the projection*

$$\pi: F(\Gamma) \to T(\Gamma) \tag{2.2.1}$$

onto the first factor with the forgetful map f_1.

The space $T(p, n+1)$ is represented by a torsion-free group Γ' of type $(p, n+1)$. We use the machinery of the last paragraph to describe the map $\psi: T(\Gamma') \to F(\Gamma)$. Let $a \in U$ with $t(a) = x_1$. For $\mu \in M(\Gamma')$, let $\psi(\mu) = (\Phi(\nu), w^{\nu}(a))$, where $\nu = h^*\mu$.

2.3. The group G acts discontinuously on $F(G)$ as a group of biholomorphic mappings by

$$g(\Phi(\mu), z) = (\Phi(\mu), g^{\mu}(z)), \tag{2.3.1}$$

where $\mu \in M(G)$, $z \in \Delta^{\mu}$, $g \in G$, and

$$g^{\mu} \circ w^{\mu} = w^{\mu} \circ g. \tag{2.3.2}$$

The quotient space $V(G) = F(G)/G$ is a complex manifold, and the natural projection $F(G) \to \hat{T}(G)$ induces a holomorphic projection

$$\pi: V(G) \to \hat{T}(G),$$

with

$$\pi^{-1}(\Phi(\mu)) = \Delta^{\mu}/G^{\mu} \quad \text{for each } \mu \in M(G).$$

If Γ is a torsion-free Fuchsian group of type (p, n), then $V(\Gamma)$ depends only on the type of Γ and it is called the *n-punctured Teichmüller curve* and is denoted by $V(p, n)'$. For each $x \in T(p, n)$, the fiber $\pi^{-1}(x)$ in $V(p, n)'$ is a Riemann surface of type (p, n). By Theorem 2.2, $T(p, n+1)$ is a holomorphic universal covering space of $V(p, n)'$. The projection of $V(p, n)'$ onto $T(p, n)$ will be denoted by π_n'.

If Γ is of type (p, n) without parabolic elements, then $V(\Gamma)$ is the *Teichmüller curve* (it is independent of the signature of Γ; that is, it depends only on the type of Γ), and it is denoted by $V(p, n)$. The corresponding projection will be denoted by π_n. For each $x \in T(p, n)$, the fiber $\pi_n^{-1}(x)$ in $V(p, n)$ is a closed Riemann surface of genus p on which n points have been distinguished. There clearly is a canonical inclusion $V(p, n)' \to V(p, n)$ that makes the following diagram

$$\begin{array}{ccc} V(p,n)' & \longrightarrow & V(p,n) \\ & {\scriptstyle \pi_n'} \searrow \quad \swarrow {\scriptstyle \pi_n} & \\ & T(p,n) & \end{array}$$

commute.

2.4. There is a map $T(p + 1, 0) \to T(p, 0)$ that corresponds to pinching off a handle. We proceed to describe this map.

Let G be a Schottky group on the $p + 1$ free generators $A_1, \ldots, A_{p+1}$ with region of discontinuity Δ. We leave it to the reader to identify $S(1)$ with $\{\lambda \in \mathbf{C}; |\lambda| > 1\}$.

Assume now that $p \geqslant 1$. Let G_0 be the subgroup of G generated by $A_1, \ldots, A_p$. The region of discontinuity of G_0 will be denoted by Δ_0. Note that $G_0 \subsetneq G$, $\Delta_0 \supsetneq \Delta$, but $\Delta_0 \setminus \Delta$ has measure zero. Hence, we can identify $L^\infty(G)$ with a subset of $L^\infty(G_0)$,

$$L^\infty(G) = \left\{ \mu \in L^\infty(G_0); (\mu \circ A_{p+1}) = \mu\left(A'_{p+1}/\overline{A'_{p+1}}\right)\right\}.$$

The natural injection $M(G) \to M(G_0)$ projects to a surjective holomorphic mapping

$$f: \hat{T}(G) \to \hat{T}(G_0). \tag{2.4.1}$$

For $\mu \in M(G)$, the differential $f'(\Phi(\mu))$ is the canonical surjective map

$$L^\infty(G^\mu)/Q(G^\mu)^\perp \to L^\infty(G_0^\mu)/Q(G_0^\mu)^\perp.$$

The dual map on the cotangent level

$$\Theta: Q(G_0^\mu) \to Q(G^\mu) \tag{2.4.2}$$

is the relative Poincaré series defined by

$$(\Theta\varphi)(z) = \sum_{g \in G^\mu/G_0^\mu} \varphi(gz)g'(z)^2, \qquad z \in \Delta^\mu, \varphi \in Q(G_0^\mu). \tag{2.4.3}$$

The mapping Θ is injective (because its dual is surjective).

THEOREM. *Let f: $\hat{T}(G) \to \hat{T}(G_0)$ be the holomorphic map of* (2.4.1). *The induced map on cotangent spaces at $\Phi(\mu)$, $\mu \in M(G)$, is the relative Poincaré series map of* (2.4.2).

We have constructed a holomorphic map $S(p + 1) \to S(p)$ that corresponds to squeezing off a handle. Since $T(p, 0)$ is a holomorphic universal covering space of $S(p)$, the above theorem together with some algebraic topology leads to the

COROLLARY. *For every $p \geqslant 1$, there is a surjective holomorphic map $T(p + 1, 0) \to T(p, 0)$ that corresponds to squeezing off a handle.*

2.5. In order to construct a map with one-dimensional fibers (a map: $T(p + 1, 0) \to T(p, 2)$, for example) we must either extend the concept of deformation spaces (in this section) or study more general function groups (next section). Choose $n \geqslant 1$ inequivalent points in Δ: $z_1, \ldots, z_n$. Introduce a new equivalence relation in $M(G)$: μ is *equivalent* to ν if and only if

$$w^{\mu}|(\Lambda \cup \Lambda_0) = w^{\nu}|(\Lambda \cup \Lambda_0),$$

where

$$\Lambda_0 = \left(\bigcup_{j=1}^{n} \{ g(z_j);\, g \in G \} \right).$$

Denote the corresponding quotient space by the symbol $\hat{T}(G;\, z_1, \ldots, z_n)$. This space is a complex analytic manifold whose universal holomorphic covering space is biholomorphically equivalent to $T(\Gamma)$, where Γ is Fuchsian model for action of G on $\Delta \setminus \Lambda_0$.

If G is a Schottky group of genus $p \geqslant 1$, the prior construction depends only on the genus p of G and the number n of points and yields $S(p, n)$, which is defined up to biholomorphic equivalence. The universal holomorphic covering space of $S(p, n)$ can be identified with $T(p, n)$. Of course, $S(p, 0) = S(p)$. Returning to the situation of §2.4, we assume that $A = A_{p+1}$ does not fix $0, 1, \infty$. Let z_1 and z_2 be the attracting and repelling fixed points of A. Note that z_1 and z_2 are points of $\Delta_0 \setminus \Delta$. We now have a well-defined surjective holomorphic mapping

$$\hat{T}(G) \to \hat{T}(G_0;\, z_1, z_2),$$

by the formula

$$\mu \mapsto (\mu, w^{\mu}(z_1), w^{\mu}(z_2)).$$

The analysis of the previous section goes through in this situation. We remark that the "correct" space of quadratic differentials $Q(G_0;\, z_1, z_2)$ consists of functions which are permitted to have simple poles at

$$\Lambda_0 = \bigcup \{ g(z_j);\, g \in G_0, j = 1, 2 \}.$$

THEOREM. *There exists for every $p \geqslant 1$ a holomorphic surjective mapping: $T(p+1, 0) \to T(p, 2)$ that corresponds to squeezing off a handle.*

The fibers of the above map are rather complicated. In particular, we do not even know if the fibers are connected.

2.6. If G is an arbitrary Kleinian group with an invariant component Δ, and G_1 is a subgroup with invariant component $\Delta_1 \supset \Delta$, then the restriction mapping induces a holomorphic mapping $\hat{T}(G) \to \hat{T}(G_1)$. For special classes of G_1 (for example, factor subgroups (see Maskit **[Ms4]** for a definition)), the above mapping is quite interesting and leads to another construction of a "handle squeezing map" $T(p+1, 0) \to T(p, 2)$. This line of investigation (joint work with Maskit) will be persued elsewhere **[KM2]**. We outline here one special case.

Let Γ be a torsion free Fuchsian group of type (p, n), $n \geqslant 2$, $2p + n > 2$, operating on the upper half-plane U. We can construct a function group G with simply-connected invariant component $\Delta \subset U$ that represents a surface of type $(p+1, n-2)$ by adjoining an element $\gamma \in \mathrm{Mob}$ to Γ. The element γ conjugates a parabolic subgroup corresponding to one puncture onto a parabolic subgroup representing a second puncture. We obtain this way a surjective map $T(p+1, n-2) \to T(p, n)$.

It is an open problem to describe the fibers of these maps as well as the relation of these maps to the ones introduced in §2.5.

2.7. If Γ is a finitely-generated Fuchsian group of the first kind, then $T(\Gamma)$ is a domain of holomorphy (Bers-Ehrenpreis **[BE]**). Hejhal **[He]** showed that for Schottky groups G, $\hat{T}(G)$ is a domain of holomorphy. The same result also holds for arbitrary groups (Kra-Maskit **[KM2]**). The proofs use Maskit's **[Ms4]** structure theorems for function groups.

3. Allowable mappings and the modular group.

3.1. Let θ: $G \to G'$ be a geometric isomorphism (see §1.12) between finitely-generated nonelementary Kleinian groups induced by the quasiconformal automorphism f: $\hat{\mathbf{C}} \to \hat{\mathbf{C}}$. The map f induces a biholomorphic map

$$f^*: M(G) \to M(G')$$

(the elements of $M(G')$ are supported on the invariant component $\Delta' = f(\Delta)$ of G') by sending the Beltrami coefficient $\nu \in M(G)$ into the ltrami coefficient of $w^\nu \circ f^{-1}|\Delta'$. If μ is the Beltrami coefficient of f, then

$$f^*(\nu) \circ f = \frac{\nu - \mu}{1 - \bar{\mu}\nu} \frac{f_z}{\overline{f_z}}. \tag{3.1.1}$$

Formula (3.1.1) shows at once that f^* is a biholomorphic mapping. Since f^* carries equivalent elements of $M(G)$ into equivalent elements of $M(G')$, it induces an *allowable* biholomorphic

$$\theta^*: \hat{T}(G) \to \hat{T}(G')$$

which depends only on the isomorphism θ, in fact only on the conjugacy class of θ modulo inner automorphisms of G and G' (that is, θ and $i(g') \circ \theta \circ i(g)$, where $i(g)$: $G \to G$ is defined by $i(g)\gamma = g \circ \gamma \circ g^{-1}$, define the same allowable map).

The allowable map θ sends $\Phi(0) \in \hat{T}(G)$ into $\Phi(\sigma) \in \hat{T}(G')$, where σ is the Beltrami coefficient of f^{-1}, and thus establishes an isometric isomorphism between the tangent (and cotangent) space of $\hat{T}(G)$ at $\Phi(0)$ and the corresponding tangent (and cotangent) space of $\hat{T}(G')$ at $\Phi(\sigma)$.

3.2. We can define the *modular group* Mod G as the quotient of the group of geometric automorphisms induced by quasiconformal maps f that preserve Δ ($f\Delta = \Delta$) by the normal subgroup of inner automorphisms. (Note that the action of Mod G on $\hat{T}(G)$ is *not* always effective.)

The *moduli space* (or Riemann space) $R(G) = \hat{T}(G)/\text{Mod } G$ is a complex normal space (not a manifold) and represents the set of conjugacy classes of Kleinian groups quasiconformally equivalent to G (see, for example, Bers **[B11]**).

3.3. If Γ is Fuchsian, Mod Γ depends only on the signature of Γ. We set Mod(p, n) to be Mod Γ for a torsion-free group Γ of type (p, n).

If X is a surface of type (p, n), then Mod(p, n) can be identified with the group of orientation-preserving automorphisms of X modulo those homotopic to the identity.

The group Mod(p, n) acts discontinuously on $T(p, n)$ (see, for example, Bers **[B11]**), and hence (as a particular special case) the Riemann space

$R(p, n) = T(p, n)/\mathrm{Mod}(p, n)$ is a normal complex space (Bers **[B11]**).

REMARK. The group $\mathrm{Mod}(p, n)$ acts effectively on $T(p, n)$ except if the type (p, n) is *exceptional*; that is, the type appears in the following short list: (0, 3), (0, 4), (1, 1), (1, 2) and (2, 0).

3.4. The group of geometric automorphisms of G induced by quasiconformal self-maps that preserve Δ is usually denoted by the symbol mod G. We clearly have the following relations

$$G \subset \operatorname{mod} G, \qquad \operatorname{Mod} G = \operatorname{mod} G/G.$$

The group mod G acts as a group of biholomorphic automorphisms of $F(G)$ as follows. If $\theta \in \operatorname{mod} G$ is induced by the quasiconformal map f, then

$$\theta(\Phi(\mu), z) = (\Phi(\nu), \hat{z}),$$

where $\mu \in M(G)$, $z \in w^{\mu}(\Delta)$, ν = Beltrami coefficient of $w^{\mu} \circ f^{-1}$, and

$$\hat{z} = w^{\nu} \circ f \circ (w^{\mu})^{-1}(z).$$

The action of mod G on $F(G)$ is always effective. The action of the subgroup G of mod G coincides with the action of G described in §2.3 by equations (2.3.1) and (2.3.2).

The action of mod G on $F(G)$ induces an action of mod G on $V(G) = F(G)/G$. The group G acts trivially on $V(G)$. Hence we have actually an action of Mod G on $V(G)$. This action is always effective, and it motivates the definition of the modular group.

3.5. Let H be a finite subgroup of $\mathrm{Mod}(p, n)$. The fixed point set $T(p, n)^H = \{t \in T(p, n);\ ht = t \text{ all } h \in H\}$ is again a Teichmüller space $T(p', n')$ provided (see §6.11) that it is nonempty **[Kz]**. Kravetz **[Kz]** claimed that this set is nonempty for all finite H. The proof in **[Kz]** has a gap. However, the arguments of that paper for the cyclic case show that the fixed point set is nonempty whenever H is a solvable group.

Quite recently S. Kerckhoff **[Ke]** has announced a proof that the fixed point set is always nonempty. His proof uses Thurston's **[Th]** theory of earthquakes.

4. Metrics on deformation spaces.

4.1. If ω: $\hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is a μ-conformal mapping, its maximal dilatation $K(\omega)$ is given by

$$K(\omega) = \frac{1 + \|\mu\|_{\infty}}{1 - \|\mu\|_{\infty}}.$$

The *Teichmüller metric* on $M(G)$ is defined by

$$d(\mu, \nu) = \tfrac{1}{2} \log K\big(w^{\mu} \circ (w^{\nu})^{-1}\big).$$

This metric d is complete and it induces the same topology as the L^{∞}-norm. It is the non-Euclidean version of the L^{∞}-metric. The *Teichmüller metric* on $\hat{T}(G)$ is the quotient metric of d

$$\tau(\Phi(\mu_0), \Phi(\nu_0)) = \inf\{d(\mu, \nu);\ \mu, \nu \in M(G),$$
$$\Phi(\mu) = \Phi(\mu_0), \Phi(\nu) = \Phi(\nu_0)\}.$$

REMARK. The deformation space $\hat{T}(G)$ is the quotient of some Teichmüller space $T(\Gamma)$ by a fixed point free subgroup of Mod Γ. The Teichmüller metric on $\hat{T}(G)$ is the quotient metric of the Teichmüller metric on $T(\Gamma)$.

4.2. A Beltrami coefficient $\mu \in M(G)$ is called *extremal* if and only if

$$\tau(\Phi(0), \Phi(\mu)) = d(0, \mu).$$

Teichmüller's theorem (see Ahlfors **[Ah2]** or Bers **[B2]**) states that if μ is extremal, then μ is a *Teichmüller differential*; that is, $\mu = 0$ or $\mu = k\bar{\varphi}/|\varphi|$, $k \in \mathbf{R}$, $0 < k < 1$, $\varphi \in Q(G)$, $\varphi \neq 0$. If Δ is simply connected (for example, if we are dealing with Fuchsian groups), then every Teichmüller differential is extremal and each $\nu \in M(G)$ is equivalent to exactly one Teichmüller differential. It follows easily from this result that for Γ Fuchsian of type (p, n), $T(\Gamma)$ is homeomorphic to $Q(\Gamma) \cong \mathbf{C}^{3p-3+n}$.

4.3. Consider a Fuchsian group Γ (including the trivial group and groups of the second kind–as well as infinitely-generated groups) operating on the upper half-plane U. Two elements $\mu, \nu \in M(\Gamma)$ are called Γ-*equivalent* if and only if $w^{\mu}|\mathbf{R} = w^{\nu}|\mathbf{R}$ (equivalently if and only if $w^{\mu}|U^* = w^{\nu}|U^*$). Here $\mathbf{R}$ is the real axis and U^* is the lower half-plane.

If Γ is of the first kind, then the concepts of Γ-equivalence–and equivalence (as introduced in §1.2) agree. For groups of the second kind (including the trivial group), Γ-equivalence implies equivalence, but not conversely.

An element $\mu \in M(\Gamma)$ is Γ-*extremal* if and only if $\|\mu\|_\infty < \|\nu\|_\infty$ all $\nu \in M(\Gamma)$ such that ν is Γ-equivalent to ν. (Again, if Γ is finitely generated of the first kind, then μ is Γ-extremal if and only if μ is extremal.) A Beltrami coefficient $\mu \in M(\Gamma)$ is Γ-extremal (Hamilton **[Hm]**, Reich-Strebel **[RS]**, and Strebel **[St1]**) if and only if μ satisfies the *Hamilton condition*; that is,

$$\|\mu\|_\infty = \sup\left\{\frac{1}{2}\left|\int\int_{U/\Gamma} \mu(z)\varphi(z)dz \wedge d\bar{z}\right|; \varphi \in Q(\Gamma), \|\varphi\| = 1\right\}.$$

The question of which Beltrami coefficients are the unique extremals in their equivalence classes is under active investigation as is a related question to be discussed later.

Let Γ' be a subgroup of Γ. Then we can ask:

If $\mu \in M(\Gamma)$ is Γ-extremal, when is it Γ'-extremal?

If Γ' is of finite index in Γ, then the answer is (trivially) always yes. Examples of $\mu \in M(\Gamma)$, Γ finitely generated of the first kind, where μ is Γ-extremal but not $\{1\}$-extremal (extremal for the trivial group) have been obtained by Strebel **[St2]**. This question is intimately related to determining the norm of the Poincaré series operator

$$\Theta\colon Q(\{1\}) \to Q(\Gamma),$$

defined as in (2.4.3) by

$$(\Theta\varphi)(z) = \sum_{g\in\Gamma} \varphi(gz)g'(z)^2, \qquad z \in U, \varphi \in Q(\{1\}).$$

If for a given group Γ, the (operator) norm of Θ, $\|\theta\Theta\|$, is less than one, then every $0 \neq \mu \in M(\Gamma)$ which is Γ-extremal cannot be $\{1\}$-extremal.

Several people are currently trying to determine $\|\Theta\|$–by direct methods as well and through the connection with quasiconformal mappings.

4.4. The Teichmüller metrics d and τ are both distance functions associated with Finsler structures. The Finsler structure α on the tangent bundle $M(G) \times L^\infty(G)$ of $M(G)$ is defined by

$$\alpha(\mu, \nu) = \left\| \frac{\nu}{1 - |\mu|^2} \right\|_\infty, \qquad \mu \in M(G), \nu \in L^\infty(G).$$

(The norm $\|\cdot\|_\infty$ is the L^∞-norm.) The induced distance is the Teichmüller metric d (Earle-Eells [**EE**]). The Finsler structure β on the tangent bundle of $\hat{T}(G)$ is the quotient of α; that is,

$$\beta(\Phi(\mu), \Phi'(\mu)\nu) = \inf\{\alpha(\mu, \nu + \lambda); \Phi'(\mu)\lambda = 0\}.$$

One must check that β is well defined (quite easy) and that β induces the Teichmüller metric τ on $\hat{T}(G)$ (O'Byrne [**O**]).

Since $\alpha(0, \cdot)$ is the usual norm on $L^\infty(G)$, the quotient norm $\beta(\Phi(0), \cdot)$ on the tangent space $Q(G)^*$ is also the usual norm, conjugate to the L^1-norm on $Q(G)$.

4.5. The *Kobayashi* (or *hyperbolic*) *pseudo-metric* [**Ko**, pp. 45–56] on a complex manifold M is the largest pseudo-metric k on M such that

$$k(f(z_1), f(z_2)) \leqslant \rho(z_1, z_2)$$

for all holomorphic maps f of the unit disk into M and for all z_1, z_2 in the unit disk. Here ρ is the Poincaré metric on the unit disk

$$\rho(z_1, z_2) = \tanh^{-1}\left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|.$$

The *Carathéodory pseudo-metric* [**Ko**, pp. 49–53] on M is the smallest pseudo-metric c on M such that

$$\rho(f(x_1), f(x_2)) \leqslant c(x_1, x_2)$$

for all holomorphic mappings of M into the unit disk and all x_1, x_2 in M.

The classical Schwarz-Pick lemma shows that $c \leqslant k$. Since both c and k are intrinsically defined, all biholomorphic maps are isometries in these two metrics.

4.6. The allowable mappings between Teichmüller spaces are isometries in the Teichmüller metrics (by formula (3.1.1)) and in the Kobayashi and Carathéodory metrics (by the intrinsic nature of these metrics). For $\mu \in M(G)$, $g \mapsto w^\mu \circ g \circ (w^\mu)^{-1}$ is a geometric isomorphism θ of G onto G^μ. Since $\theta^{*-1}\colon \hat{T}(G^\mu) \to \hat{T}(G)$ sends $\Phi(\sigma)$ onto $\Phi(0)$ (where σ = Beltrami coefficient of $(w^\mu)^{-1}$), we see that in all calculations of distances between points of a deformation space (or the space of Beltrami coefficients), we may take one of the points to be the origin. Further, if we consider, for example, a Fuchsian group Γ, and an arbitrary point $x \in T(\Gamma)$, then an allowable map θ^* takes $T(\Gamma)$ onto another Teichmüller space $T(\Gamma')$ with $\theta^*(x)$ = origin in $T(\Gamma')$.

4.7. The Hahn-Banach theorem along with the considerations of the previous paragraph, show that $c = k = d$ on $M(G)$. The fact that τ is the quotient metric of d shows that

$$c \leqslant k \leqslant \tau \quad \text{on } \hat{T}(G). \tag{4.7.1}$$

Teichmüller's theorem shows that for a Fuchsian group Γ and $\varphi \in Q(\Gamma)$, $\varphi \neq 0$, $z \mapsto \Phi(z\bar{\varphi}/|\varphi|)$ is an isometric mapping of the unit disk into the Teichmüller space $T(\Gamma)$ with the τ-metric. It is very tempting to

CONJECTURE. *On* $T(p, n)$, $c = k = \tau$.

To establish this conjecture, one has to produce for fixed $\varphi \in Q(\Gamma)$, $\varphi \neq 0$, a holomorphic function f: $T(p, n) = T(\Gamma) \to$ (unit disk) so that

$$\left| f\left(z_0 \frac{\bar{\varphi}}{|\varphi|} \right) \right| = |z_0| \quad \text{for some } z_0 \neq 0, |z_0| < 1.$$

This can be done if, for example, $n = 0$ and φ has only even order zeros (Kra **[Kr5]**). One can also obtain some other qualitative results about the metric c (Earle **[E2]**, Krushkal **[Ku]**, Kra **[Kr4]**); for example, it is complete. However, the conjecture is still an open problem.

Positive evidence towards the conjecture is provided by the following beautiful and important

THEOREM (ROYDEN **[Ro]**). *The Kobayashi metric of* $\hat{T}(G)$ *is the Teichmüller metric. In fact,*

$$\tau(x_1, x_2) = \inf\{\rho(z_1, z_2); f\colon (\text{unit disk}) \to \hat{T}(G) \textit{ is holomorphic}, f(z_j) = x_j, j = 1, 2\}.$$

Royden **[Ro]** only proved the theorem for the Teichmüller spaces $T(p, 0)$, $p \geqslant 2$. The result for $T(p, n)$ can be obtained by realizing that all the arguments used carry over with no change to the more general situation. The extension can, however, also be obtained by the following simple argument that is useful in many other situations. Represent $T(p, n)$ by $T(\Gamma)$ where Γ has no parabolic elements. By Selberg's **[Se]** theorem, Γ has a fixed point free normal subgroup Γ' of finite index. Now $T(\Gamma')$ represents $T(p', 0)$ for some $p' > 2$. The natural injective holomorphic map i: $T(\Gamma) \to T(\Gamma')$ is distance nonincreasing in the Kobayashi metric and distance preserving in the Teichmüller metric. Thus, for all $x_1, x_2 \in T(\Gamma)$,

$$\tau(x_1, x_2) = \tau(i(x_1), i(x_2)) = k(i(x_1), i(x_2)) \leqslant k(x_1, x_2).$$

This inequality combined with (4.7.1) yields the required extension. Since both the Kobayashi and Teichmüller metrics project to factor spaces (obtained by actions of fixed point free discrete groups), it is easy to extend the theorem to $\hat{T}(G)$ (see Gentilesco **[Gn]**).

COROLLARY 1. *If* f: $\hat{T}(G) \to \hat{T}(G')$ *is holomorphic then*

$$\tau(f(x_1), f(x_2)) \leqslant \tau(x_1, x_2) \quad \textit{all } x_1, x_2 \in \hat{T}(G).$$

COROLLARY 2. *If* f: $\hat{T}(G) \to \hat{T}(G')$ *is biholomorphic, then* f *is an isometry.*

Since τ is the metric induced by the Finsler structure β, Corollary 1 has the following infinitesimal version.

COROLLARY 3. *A holomorphic mapping* f: $\hat{T}(G) \to \hat{T}(G')$ *cannot increase the* β*-length of any vector.*

Since the dual metric on the cotangent bundle is well understood, we also have

COROLLARY 4. *Let* F: $\hat{T}(G) \to \hat{T}(G')$ *be holomorphic. Let* $x \in \hat{T}(G)$ *be represented by the Kleinian group* G_1, *and let* $y = f(x) \in \hat{T}(G')$ *be represented by the Kleinian group* G_2. *Let* L: $Q(G_2) \to Q(G_1)$ *be the map induced between the cotangent space to* $\hat{T}(G')$ *at* y *and the cotangent space to* $\hat{T}(G)$ *at* x. *Then*

$$\|L\varphi\| \leqslant \|\varphi\| \quad \textit{all } \varphi \in Q(G_2).$$

REMARK. The above considerations also show that it suffices to establish the conjecture only for $T(p, 0), p \geqslant 2$.

4.8. Corollary 4 becomes extremely useful when combined with very precise information about the smoothness of the unit sphere in $Q(G)$. The following lemma, which appeared in Earle-Kra [**EK1**], is a straightforward generalization of a result of Royden [**Ro**, Lemma 1].

Let φ and ψ be L^1-functions on the unit disk Δ, holomorphic and nonzero for $z \neq 0$, and bounded except possibly in a neighborhood of $z = 0$. Let

$$\nu = \operatorname{ord}_0 \varphi, \qquad \mu = \operatorname{ord}_0 \psi$$

(note that $\mu, \nu \geqslant -1$). Define for real t

$$f(t) = \int\int_{\Delta} |\varphi(z) + t\psi(z)|\, |dz \wedge d\bar{z}|.$$

LEMMA. *The function* f *is a differentiable function of* t *near* $t = 0$, *and*

$$f'(0) = \int\int_{\Delta} \operatorname{Re}[\psi(z)\bar{\varphi}(z)/|\varphi(z)|]|dz \wedge d\bar{z}|.$$

Furthermore, if $\mu \geqslant (\nu - 1)/2$, *then* f *has a second derivative at* $t = 0$ *given by*

$$f''(0) = \int\int_{\Delta} |\varphi(z)|^{-3}[\operatorname{Im} \psi(z)\bar{\varphi}(z)]^2 |dz \wedge d\bar{z}|.$$

If $\mu < (\nu - 1)/2$, *then*

$$f(t) = f(0) + tf'(0) + c\varepsilon(t) + o(\varepsilon(t))$$

with $c > 0$, *where*

$$\begin{aligned} \varepsilon(t) &= t^2 \log(1/|t|) \quad \textit{if } \nu = 2\mu + 2, \\ &= |t|^{1 + [(2+\mu)/(\nu - \mu)]} \quad \textit{if } \nu > 2\mu + 2. \end{aligned}$$

REMARK. The previous lemma shows that the L^1-metric on the cotangent space (with the zero section removed) is of class C^1 but not C^2. Therefore, the β-metric (the Finsler structure on the tangent space dual to the L^1-metric on the cotangent space) is also C^1 (see Royden [**Ro**]). Earle [**E4**] has shown that the Teichmüller metric τ (the integrated form of the β-metric) is also C^1 off the diagonal.

4.9. We must also describe some metrics on the fiber spaces $V(p, n)'$. Again, it is easiest to describe the metrics on the cotangent level. The

Kobayashi metric behaves well under covering mappings, Kobayashi [**Ko**, p. 48]. Since we know the holomorphic universal covering space of $V(p, n)'$ to be $T(p, n+1)$, we easily arrive at

PROPOSITION (EARLE-KRA [**EK3**]). *Let π_n': $V(p, n)' \to T(p, n)$ be the punctured Teichmüller curve of type (p, n), $2p + n > 2$. Let $x_0 \in V(p, n)'$ and $\tau_0 = \pi_n'(x_0) \in T(p, n)$. Let $X = \pi_n'^{-1}(\tau_0)$. Then the cotangent space to $T(p, n)$ at τ_0 is $Q(X)$, and the cotangent space to $V(p, n)'$ at x_0 is $Q(X \setminus \{x_0\})$. The map of cotangent spaces induced by π_n' is the inclusion of $Q(X)$ in $Q(X \setminus \{x_0\})$. Further, the Finsler metric on $V(p, n)'$ induced by the norm*

$$\|\varphi\| = \frac{1}{2}\int_X |\varphi|, \qquad \varphi \in Q(X \setminus \{x_0\}),$$

is the Kobayashi metric on $V(p, n)'$.

PROBLEM. Describe the Kobayashi metric on $V(p, n)$. (The problem is open for $n > 0$.)

4.10. The Teichmüller metric on $T(p, n)$ is a complete metric invariant under the modular group $\mathrm{Mod}(p, n)$. As a domain in complex number space, $T(p, n)$ also carries a Bergmann metric (see for example, [**KO**, pp. 17–19]), that is also invariant under $\mathrm{Mod}(p, n)$. The Teichmüller metric is known *not* to be Hermitian. Because the Carathéodory metric is complete, so is the Bergman metric (Hahn [**Ha1**] and [**Ha2**]). It is also Kahlerian.

There also exists another Hermitian metric on $T(p, n)$ invariant under $\mathrm{Mod}(p, n)$, the so called *Weil-Petersson metric*. This is the metric induced by the Weil-Petersson inner product.

Let $T(p, n) = T(\Gamma)$. Then the Weil-Petersson metric at $\Phi(\mu)$, $\mu \in M(\Gamma)$, is induced by the Hermitian inner product

$$\langle \varphi, \psi \rangle = \frac{1}{2}\int\int_{U^\mu/\Gamma^\mu} \lambda_\mu^{-2}(z)\varphi(z)\overline{\psi(z)}\,|dz \wedge d\bar{z}|,$$

on $Q(\Gamma^\mu)$. Here λ_μ is the Poincaré metric on U^μ.

It was shown (for $n = 0$) that this metric is Kahlerian (Weil [**We**], Ahlfors [**Ah4**]), that holomorphic sections have negative curvature (Ahlfors [**Ah5**]), and that there is a constant $c_{p,n}$ such that the Weil-Petersson distance on $T(p, n)$ does not exceed $c_{p,n}$ times the Teichmüller distance (Linch [**Li**]). It was conjectured (Bers [**B10**]) that the Weil-Petersson metric agrees with the Bergman metric and is, hence, complete. Wolpert [**Wo**] and Chu [**C**] proved that the Weil-Petersson metric is not complete (for $n = 0$). Further, Masur [**Mr2**] showed that the Weil-Petersson metric can be extended to $T(p, 0)$ with all its boundary spaces, and that it projects to a complete metric on the compactified moduli space.

5. Maps between Teichmüller spaces.

5.1. We have already remarked that an inclusion of Fuchsian groups $\Gamma \hookrightarrow \Gamma'$ induces an isometric embedding of $T(\Gamma')$ in $T(\Gamma)$. Hence, $T(\Gamma') = T(\Gamma)$ if and only if $\dim T(\Gamma') = \dim T(\Gamma)$. This phenomenon occurs in a few cases (Greenberg [**Gr**] or Singerman [**Si**]) and leads to three isomorphisms,

$$T(2, 0) \cong T(0, 6), \qquad T(1, 2) \cong T(0, 5), \qquad T(1, 1) \cong T(0, 4).$$

5.2. The fact that $\mathrm{Mod}(p, n)$ acts discontinuously on $T(p, n)$ has as a consequence the following

LEMMA. *Let θ: $T(p, n) \to T(p, n)$ be biholomorphic. If for each $x \in T(p, n)$ there exists a $\gamma_x \in \mathrm{Mod}(p, n)$ such that $\theta(x) = \gamma_x(x)$, then $\theta \in \mathrm{Mod}(p, n)$.*

The lemma gets its value from the well-known fact that $x, y \in T(p, n)$ represent conformally equivalent Riemann surfaces if and only if there exists a $\gamma \in \mathrm{Mod}(p, n)$ such that $\gamma(x) = y$. Thus, to prove that an automorphism θ of $T(p, n)$ is an element of the modular group, it suffices to show that for each $x \in T(p, n)$, $\theta(x)$ represents a Riemann surface conformally equivalent to the surface represented by x.

5.3. Our aim is to describe Aut $T(p, n)$, the group of holomorphic self-mappings of the Teichmüller space $T(p, n)$. There is another related problem: to describe Aut $V(p, n)$. This turns out to be a rather simple problem. Consider the projection

$$\pi_n\colon V(p, n) \to T(p, n).$$

Let $f \in$ Aut $V(p, n)$. For $x \in T(p, n)$,

$$\pi_n \circ f\colon \pi_n^{-1}(x) \to T(p, n)$$

is a holomorphic function from a compact Riemann surface into a bounded domain. Hence, $f(\pi_n^{-1}(x)) = \pi_n^{-1}(\theta(x))$ for some $\theta \in \mathrm{Mod}(p, n)$. We have seen in §3.4 how to extend the action of $\mathrm{Mod}(p, n)$ on $T(p, n)$ to an action on $V(p, n)$ that commutes with the projection π_n. It follows that $f \in \mathrm{Mod}(p, n)$. For details see Duma [**D**].

The description of Aut $F(\Gamma)$ is probably obtainable by methods to be described later (although this has not been done yet). If Γ is a torsion free Fuchsian group of type (p, n), then $F(\Gamma)$ is isomorphic to $T(p, n + 1)$ (Bers [**B11**]) and hence $\mathrm{Aut}(F(\Gamma))$ is known. In general (if Γ has torsion), $F(\Gamma)$ is not isomorphic to a Teichmüller space (Earle-Kra [**EK1**]), and thus it is conceivable that interesting new groups will appear in a description of Aut $F(\Gamma)$.

5.4. Let S be a Riemann surface of type (p, n). We have seen that $\mathrm{Mod}(p, n)$ is the group of sense-preserving homeomorphisms of S onto itself modulo those homotopic to the identity. The *extended modular group* $\mathrm{Mod}(p, n)^{\sim}$ is the group of all homeomorphisms of S onto itself modulo those homotopic to the identity. The group $\mathrm{Mod}(p, n)^{\sim}$ acts as a group of biholomorphic and antiholomorphic isometries of $T(p, n)$. The action is an obvious extension of the action of $\mathrm{Mod}(p, n)$ discussed in §3.2. Clearly $\mathrm{Mod}(p, n)^{\sim}/\mathrm{Mod}(p, n) \cong \mathbf{Z}_2$. Thus $\mathrm{Mod}(p, n)^{\sim}$ acts discontinuously on $T(p, n)$. (It is quite useful to think of $\mathrm{Mod}(p, n)^{\sim}$ as $\mathrm{Mod}(p, n)$ extended by an element of order 2 that sends each Riemann surface to its mirror image.)

5.5. It is slightly surprising that one needs no smoothness assumption on an isometry to obtain the following

THEOREM (EARLE-KRA [**EK2**]). *Let $T(p, n)$ and $T(p', n')$ be two Teichmüller spaces with $2p + n > 4$ and $2p' + n' > 4$. If U is a domain in $T(p, n)$ and $f: U \to T(p', n')$ is an isometry in the Teichmüller metric with open range, then $(p, n) = (p', n')$ and f is the restriction to U of an element of the extended modular group* $\mathrm{Mod}(p, n)^{\sim}$. *Thus, f is either holomorphic or conjugate holomorphic.*

The theorem has many easy consequences.

COROLLARY 1. *Every isometry of an open connected subset of $T(p, n)$ onto an open subset of $T(p', n')$ is the restriction of a global surjective isometry.*

Here and in the next corollary, we assume only that $2p + n > 2$ and $2p' + n' > 2$.

COROLLARY 2. *If $T(p, n)$ and $T(p', n')$ are locally isometric at even one point, then $T(p, n)$ is biholomorphically equivalent to $T(p', n')$.*

In the next corollary, we let $\mathrm{Isom}\, T(p, n)$ be the group of isometries of $T(p, n)$ onto itself.

COROLLARY 3. *We have*

$$\begin{aligned}
&\mathrm{Isom}\, T(p, n) = \mathrm{Mod}(p, n)^{\sim} \quad \textit{provided}\ 2p + n > 4,\\
&\mathrm{Isom}\, T(2, 0) \cong \mathrm{Mod}(0, 6)^{\sim} \cong \mathrm{Mod}(2, 0)^{\sim}/\mathbf{Z}_2,\\
&\mathrm{Isom}\, T(1, 2) \cong \mathrm{Mod}(0, 5)^{\sim},\\
&\mathrm{Isom}\, T(1, 1) \cong \mathrm{Isom}\, T(0, 4) \cong \mathrm{Mob}_{\mathbf{R}}^{\sim}, \textit{ and}\\
&\mathrm{Isom}\, T(0, 3) = \{\mathrm{id}\}.
\end{aligned}$$

In the above, $\mathrm{Mob}_{\mathbf{R}}^{\sim}$ stands for the extended real Mobius group; that is, the mappings of the upper half-plane of the form

$$z \mapsto \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbf{R}, ad - bc = 1,$$
$$z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d}, \qquad a, b, c, d \in \mathbf{R}, ad - bc = -1.$$

5.6. Since every biholomorphic map between Teichmüller spaces is an isometry in the Teichmüller metric (by Royden's [**Ro**] theorem (4.7, above)), the previous result has the following consequences. In the first theorem $\mathrm{Aut}\, T(p, n)$ denotes, as before, the group of biholomorphic self-maps of $T(p, n)$.

THEOREM (ROYDEN [**Ro**]). *We have*

$$\begin{aligned}
&\mathrm{Aut}\, T(p, n) = \mathrm{Mod}(p, n) \quad \textit{provided}\ 2p + n > 4,\\
&\mathrm{Aut}\, T(2, 0) \cong \mathrm{Mod}(0, 6) \cong \mathrm{Mod}(2, 0)/\mathbf{Z}_2,\\
&\mathrm{Aut}\, T(1, 2) \cong \mathrm{Mod}(0, 5),\\
&\mathrm{Aut}\, T(1, 1) \cong \mathrm{Aut}\, T(0, 4) \cong \mathrm{Mob}_{\mathbf{R}}, \textit{ and}\\
&\mathrm{Aut}\, T(0, 3) = \{\mathrm{id}\}.
\end{aligned}$$

We have seen that the Teichmüller space $T(p, n)$ is a holomorphic covering space of the moduli space $R(p, n)$. Are there any naturally defined complex spaces that are covered by $R(p, n)$? One way to obtain such a space is to factor $T(p, n)$ by a group Γ with $\mathrm{Mod}(p, n) \subsetneq \Gamma \subset \mathrm{Aut}\, T(p, n)$. Royden's theorem shows that such groups Γ do not exist in general.

There are only a finite number of Teichmüller spaces of a given dimension. The classification of these spaces is given by the following

THEOREM (PATTERSON [**Pa**]). *If $T(p, n)$ is biholomorphically equivalent to $T(p', n')$ and $2p + n > 4$ and $2p' + n' > 4$, then $(p, n) = (p', n')$.*

5.7. To prove Theorem 5.5, let f be an isometry from a domain U in $T(p, n)$ onto a domain U' in $T(p', n')$. Since $T(p, n)$ is biholomorphically equivalent (§1.5) to a bounded domain in $\mathbf{C}^{3p-3+n}$, and the Teichmüller metric is induced from a Finsler structure, the Teichmüller distance and the Euclidean distance (coming from the usual norm on $\mathbf{C}^{3p-3+n}$) are locally Lipschitz with respect to each other. Hence the map f is locally Lipschitz, viewed as a map from an open set in $\mathbf{C}^{3p-3+n}$ into $\mathbf{C}^{3p-3+n}$. By the Rademacher-Stepanov theorem (see, for example, Väisälä [**V**, p. 97] or Federer [**Fe**, p. 216]), f is differentiable almost everywhere.

Let f be differentiable at $x \in U$. Then f induces a **R**-linear isometry between the cotangent space to $T(p, n)$ at x and the cotangent space to $T(p', n')$ at $f(x)$. If we could show (see §5.2) that this implies that x and $f(x)$ represent either conformally equivalent or anticonformally equivalent Riemann surfaces, we would be done by Lemma 5.2 (which can be generalized to apply to isometries and elements of the extended modular group).

5.8. To study isometries between integrable holomorphic quadratic differentials with L^1-norm, we make use of Lemma 4.8 and establish the following results.

PROPOSITION (EARLE-KRA [**EK2**]). *Let X and X' be Riemann surfaces of type (p, n) and (p', n'), respectively. Assume $2p + n > 4$ and $2p' + n' > 4$.*

(a) *If $A\colon Q(X) \to Q(X')$ is a surjective* **R**-*linear isometry, then A is either* **C**-*linear or conjugate* **C**-*linear.*

(b) *Further, $(p, n) = (p', n')$.*

The proposition when combined with the following theorem completes this circle of ideas.

THEOREM (ROYDEN [**Ro**], EARLE-KRA [**EK1**]). *Let X and X' be Riemann surfaces of finite type (p, n) and (p', n') with $2p + n > 4$ and $2p' + n' > 4$. Then every* **C**-*linear isometry $A\colon Q(X) \to Q(X')$ is of the form $A(\varphi) = \alpha f^*\varphi$, where $\alpha \in \mathbf{C}$, $|\alpha| = 1$, $f\colon X' \to X$ is a conformal map, and $f^*\varphi$ is the pullback of φ by f.*

The theorem shows that the only isometries between spaces of integrable holomorphic quadratic differentials are the ones arising quite naturally.

5.9. The prior results can now be generalized to the deformation spaces $\hat{T}(G)$ and their generalizations. One can show (see Gentilesco [**Gn**]) that we have the following result.

THEOREM. *The Teichmüller spaces $T(p, n)$ are not biholomorphically equivalent to products of lower-dimensional Teichmüller spaces $T(p_1, n_1) \times T(p_2, n_2)$ for all (p, n) with $2p + n - 2 > 0$.*

6. Sections of families of closed surfaces.

6.1. We return to the Teichmüller curve π_n: $V(p, n) \to T(p, n)$ represented by a concrete Fuchsian group Γ without parabolic elements

$$\pi: V(\Gamma) \to T(\Gamma). \tag{6.1.1}$$

If $z_0 \in U$ is a fixed point of $\gamma \in \Gamma$, then for all $\mu \in M(\Gamma)$, $w^\mu(z_0)$ is the corresponding fixed point in the fiber $w^\mu(U)$ in $F(\Gamma)$ of the element $\gamma^\mu \in \Gamma^\mu$. The map

$$\Phi(\mu) \mapsto (\Phi(\mu), w^\mu(z_0)) \tag{6.1.2}$$

is a well-defined holomorphic section of (2.2.1), and it projects to a section of (6.1.1). These are the so-called *canonical holomorphic sections*

$$s_j: T(p, n) \to V(p, n).$$

It is quite easy to see that

$$V(p, n)' = V(p, n) \setminus \bigcup_{j=1}^{n} s_j(T(p, n)).$$

6.2. There are a few obvious sections of

$$\pi_n': V(p, n)' \to T(p, n). \tag{6.2.1}$$

Assume Γ has no elliptic elements and that it has type (2, 0) or (1, 2). Then (as we have already remarked in §5.1), there is a group Γ' which contains Γ as a subgroup of index two. Further, $T(\Gamma') = T(\Gamma)$. Then $F(\Gamma') = F(\Gamma)$, and formula (6.1.2) produces sections of π_n'. These are the *Weierstrass sections.*

THEOREM (HUBBARD **[Hu1]** AND **[Hu2]**, EARLE-KRA **[EK1]**). (a) *If* $\dim T(p, n) > 1$, *the only holomorphic sections of* (6.2.1) *are the Weierstrass sections, which occur for* $(p, n) = (2, 0)$ *or* $(1, 2)$.

(b) *If* $\dim T(p, n) \leq 1$, *then for each* $x \in V(p, n)'$ *there is a unique holomorphic section s of* (6.2.1) *with* $s(\pi_n'(x)) = x$.

6.3. Represent $T(p, n)$ by $T(\Gamma)$, with Γ fixed point free. By Teichmüller's theorem, $T(p, n)$ is contractible. Thus, every section of (6.1.1) lifts to the universal covering space $T(p, n + 1)$ of $V(p, n)'$ (which can be represented by $T(\Gamma')$, with Γ' fixed point free) producing a section s of the projection $f_1 = \pi$ of (2.1.1). Further, we may choose Γ and Γ' so that $s(0) = 0$. The map f: $T(\Gamma') \to T(\Gamma')$ defined by $f = s \circ \pi$ is holomorphic, fixes zero, and satisfies $f = f \circ f$. Let P: $Q(\Gamma') \to Q(\Gamma')$ be the induced linear map on cotangent vectors to $T(\Gamma')$ at zero. Then,

$$P^2 = P, \tag{6.3.1}$$

$$\|P\| \leq 1, \tag{6.3.2}$$

and

$$\text{range } P = \text{range } \pi'(0)^* = Q(\Gamma). \tag{6.3.3}$$

6.4. The proof of Theorem 6.2 depends almost entirely on studying the projections P of §6.3, and showing that, in general, such a P cannot exist. Such projections are best studied on the Riemann surfaces. Let X be a surface of type (p, n) and let $X' = X \setminus \{x_0\}$ where $x_0 \in X$. Thus, X' has type $(p, n+1)$. We can, of course, choose X and X' so that $X = U/\Gamma$ and $X' = U/\Gamma'$, and hence, $Q(X) = Q(\Gamma)$, $Q(X') = Q(\Gamma')$. We now have $Q(X) \hookrightarrow Q(X')$, and we have already seen in §2.1 that the range of $\pi'(0)^*$ of §6.3 can be identified with $Q(X)$. Hence, we can view P as a projection $P\colon Q(X') \to Q(X)$.

LEMMA. *There is, at most, one* $P\colon Q(X') \to Q(X)$ *satisfying* (6.3.1)–(6.3.3).

6.5. In a few instances, we can actually produce such a projection. Let X be a Riemann surface of type (2, 0) or (1, 2), and let $j\colon X \to X$ be the "hyperelliptic" involution on X. Let $x_0 \in X$ be a fixed point of j. Then it is well known that $j^*\varphi = \varphi$ for all $\varphi \in Q(X)$, whereas for $\varphi \in Q(X') \setminus Q(X)$, $j^*\varphi = -\varphi + \psi$ for some $\psi \in Q(X)$. Thus, we define $P = \frac{1}{2}(I + j^*)$, where I is the identity map. Note that P satisfies (6.3.1) because $j^* = I$ on $Q(X)$. Since $\|I\| = \|j^*\| = 1$, $\|P\| = 1$. Finally, (6.3.3) is obvious.

PROPOSITION. *If* $\dim Q(X) \geqslant 2$, *then there are no projections* $P\colon Q(X') \to Q(X)$ *which satisfy* (6.3.1)–(6.3.3) *except in the situations described previously.*

The proposition is proven by the use of Lemma 4.8, relying on precise knowledge about the structure of the divisors of zeros and poles of elements of $Q(X')$. See, for example, Earle-Kra **[EK1]**.

REMARK. See **[EK1]** for a proof of Theorem 6.2(b) which was announced previously by Earle **[E1]**. The reformulation of this special case was discussed in §0.8.

6.6. There is an intrinsic description of the Weierstrass sections of §6.2. The fiber space $V(2, n)$ has a unique involution J that commutes with the projection $\pi_n\colon V(2, n) \to T(2, n)$. The restriction of J to a fiber of $V(2, n)$ is the hyperelliptic involution j of the fiber, discussed in §6.5 (recall that each surface of genus two is hyperelliptic). The fixed point set of J consists of 6 connected, closed, complex submanifolds of $V(2, n)$. The restriction of π_n to each of these submanifolds is biholomorphic onto $T(2, n)$; for $n = 0$, its inverse is one of the Weierstrass sections previously defined. In general $(n > 0)$, these will also be called *Weierstrass sections*.

There is, of course, no canonical involution on $V(1, n)$. However, on any torus there exists a unique involution taking a prescribed point onto another prescribed point. Hence $V(1, 2)'$ has a canonical involution that once again selects the "Weierstrass points".

6.7. The prior considerations have provided a complete description of the holomorphic sections of the punctured Teichmüller curve. For the case of the Teichmüller curve we have

THEOREM (EARLE-KRA **[EK3]**). *The Teichmüller curve* $\pi_n\colon V(p, n) \to T(p, n)$ *has exactly* n *holomorphic sections if* $p \geqslant 3$ *and exactly* $2n + 6$ *sections if* $p = 2$.

The n sections for $p \geqslant 3$ are the canonical sections. For $p = 2$, there are the six Weierstrass sections described in §6.6. In addition, there are the canonical sections $s_1, \ldots, s_n$ and $J \circ s_1, \ldots, J \circ s_n$, where J is the involution described in §6.6.

6.8. The proof of Theorem 6.7 proceeds in a slightly indirect fashion because we do not have a description of the Kobayashi metric on $V(p, n)$ for $n > 0$ (we do not know its universal holomorphic covering space, for example).

However, we do know the Kobayashi metric on $V(p, 0)$. To exploit this fact, we extend the forgetful map f_n: $T(p, n) \to T(p, 0)$. One can construct a holomorphic mapping F_n: $V(p, n) \to V(p, 0)$ so that the following diagram

$$\begin{array}{ccc} V(p, n) & \xrightarrow{F_n} & V(p, 0) \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_0} \\ T(p, n) & \xrightarrow{f_n} & T(p, 0) \end{array}$$

commutes. With the aid of this construction, one can establish the following

PROPOSITION. *The holomorphic sections s: $T(p, n) \to V(p, n)$ of π_n are in bijective correspondence with the holomorphic maps h: $T(p, n) \to V(p, 0)$ such that $\pi_0 \circ h = f_n$.*

6.9. Once again, the study of these maps proceeds on the cotangent level. With the identification of cotangent spaces given in §4.9, the nonexistence of holomorphic mappings h of Proposition 6.8 is equivalent to establishing the following

THEOREM (EARLE-KRA [EK3]). *Let X be a closed Riemann surface of genus $p \geqslant 2$, $X' = X \setminus \{x_0\}$, and $X'' = X \setminus \{y_1, \ldots, y_n\}$, $x_0 \in X$, and $y_1, \ldots, y_n$ are $n \geqslant 1$ distinct points in X. Let L: $Q(X') \to Q(X'')$ be a* **C***-linear map such that*

$$L\varphi = \varphi \quad \text{all } \varphi \in Q(X), \qquad \|L\varphi\| \leqslant \|\varphi\| \quad \text{all } \varphi \in Q(X').$$

If $p \geqslant 3$, then $x_0 = y_k$ for some k, and $L\varphi = \varphi$ all $\varphi \in Q(X')$. If $p = 2$, let j: $X \to X$ be the hyperelliptic involution of X. Then either $x_0 = y_k$ for some k, $j(x_0) = y_k$ for some k, or x_0 is a Weierstrass point of X.

6.10. In order to use Proposition 6.5 to prove the theorem, it is useful to establish the following two general propositions about projection operators of norm one.

Let V be any *real* Banach space whose norm $\|\cdot\|$ is a differentiable function on $V \setminus \{0\}$, and set

$$A(v, w) = \lim_{t \to 0} \frac{\|v + tw\| - \|v\|}{t}, \qquad v \in V \setminus \{0\}, w \in V.$$

Then for all $v \in V \setminus \{0\}$, $w \mapsto A(v, w)$ is a bounded linear functional on V.

PROPOSITION I. *Let W be a nontrivial closed subspace of V, and let W' be the closed subspace*

$$W' = \{v \in W;\, A(w, v) = 0 \text{ all } w \in W \setminus \{0\}\}.$$

There exists a projection P of norm one from V onto W if and only if W' is a complementary subspace to W. Further, if P exists, it is unique and its kernel is W'.

PROPOSITION II. *Let $L\colon V \to V$ be a linear map of norm one. If V has finite dimension, and the (closed) subspace $W = \{v \in V;\, Lv = v\}$ is nontrivial, then there is a projection P of norm one from V onto W.*

The proofs of the propositions are quite easy. They may be found in **[EK3]**. These two results allow us to imply from the existence of an operator L as in Theorem 6.9, a projection P as in Proposition 6.5.

6.11. Let X be a closed Riemann surface of genus $p \geqslant 2$, and let H be a nontrivial (finite) group of conformal automorphisms of X. The group H acts in a natural way (recall §3.3) on the Teichmüller space $T(p, 0)$ as a group of biholomorphic maps. The fixed point set $T(p, 0)^H$ represents Riemann surfaces of genus p which admit H as a group of automorphisms and is biholomorphically equivalent (by Kravetz **[Kz]**, for example) to $T(p', n')$, where p' is the genus of closed surface X/H and n' is the number of points in X/H over which the projection from X to X/H is branched.

The group H also acts as a group (see, for example, Bers **[B11]** or §3.4) of fiber-preserving holomorphic mappings of $V(p, 0)$. Further, each fiber is mapped onto itself. Methods similar to those outlined previously yield the following

THEOREM. *Let $s\colon T(p, 0)^H \to V(p, 0)$ be a holomorphic section of $\pi_0\colon V(p, 0) \to T(p, 0)$. If $2p' + n' > 4$, then $s(\tau)$ is fixed by a nontrivial element of H for all $\tau \in T(p, 0)^H$.*

7. Families of Jacobian varieties.

7.1. We now consider the universal Teichmüller curve $\pi_0\colon V(p, 0) \to T(p, 0)$, $p \geqslant 2$, represented by choosing a fixed Fuchsian group Γ. We choose next a canonical set of generators for Γ,

$$A_1, A_2, \ldots, A_p, B_1, B_2, \ldots, B_p; \tag{7.1.1}$$

that is, Γ is the group on the $2p$ generators listed previously subject to the single defining relation

$$\prod_{j=1}^{p} A_j B_j A_j^{-1} B_j^{-1} = 1.$$

For any $\mu \in M(\Gamma)$, the loops on the Riemann surface $w^\mu(U)/\Gamma^\mu$ determined by the prior generators have intesection numbers

$$A_j \cdot A_k = 0 = B_j \cdot B_k, \qquad A_j \cdot B_k = \delta_{jk}, \qquad 1 \leqslant j, k \leqslant p. \tag{7.1.2}$$

Let $t = \Phi(\mu)$ be a coordinate on the Teichmüller space $T(p, 0)$. Bers **[B3]** has produced holomorphic functions α_j, $1 \leqslant j \leqslant p$, on $F(\Gamma)$ satisfying

$$\alpha_j(t, \zeta) = \alpha_j(\gamma(t, \zeta))\frac{\partial\gamma}{\partial z}(t, \zeta) \quad \text{all } \gamma \in \Gamma, \tag{7.1.3}$$

and

$$\int_{\zeta}^{A_k^t(\zeta)} \alpha_j(t, \tilde{\zeta})\, d\tilde{\zeta} = \delta_{jk}, \tag{7.1.4}$$

for all $t \in T(p, 0)$, all $\zeta \in U(t) = w^{\mu}(U)$, and $1 \leqslant j,k \leqslant p$. The integral in (7.1.4) and other such integrals are to be computed along any path in $U(t)$ from ζ to $A_k^t(\zeta)$. (Recall that $A_k^t = w^{\mu} \circ A_k \circ (w^{\mu})^{-1}$ as defined in §2.3.) Formulae (7.1.3) and (7.1.4) mean that the functions $\alpha_1, \ldots, \alpha_p$ on $U(t)$ are the lifts from $X^t = U(t)/\Gamma^t$ of the normalized abelian differentials of the first kind dual to the canonical homology basis defined by (7.1.1) and (7.1.2). The *Riemann period matrix* $\tau(t) = (\tau_{jk}(t))$ of X^t is the $p \times p$ matrix with entries

$$\tau_{jk}(t) = \int_{\zeta}^{B_k^t(\zeta)} \alpha_j(t, \tilde{\zeta})d\tilde{\zeta}, \tag{7.1.5}$$

where ζ is any point in $U(t)$. The matrix τ is symmetric with positive-definite imaginary part.

7.2. Let I be the $p \times p$ identity matrix. The columns of the $p \times 2p$ matrix $(I, \tau(t))$ are linearly independent over $\mathbf{R}$ (the reals) for fixed $t \in T(p, 0)$. We introduce an action of $\mathbf{Z}^{2p}$ on $T(p, 0) \times \mathbf{C}^p$ as follows (here $\mathbf{Z}$ = integers):

$$N \cdot (t, z) = (t, z + (I, \tau(t))N) \quad \text{all } (t, z) \in T(p, 0) \times \mathbf{C}^p, N \in \mathbf{Z}^{2p}.$$

The action we have introduced is free and properly discontinuous, and $\mathbf{Z}^{2p}$ acts as a group of biholomorphic automorphisms of $T(p, 0) \times \mathbf{C}^p$.

Let $J(V(p, 0))$ be the quotient manifold,

$$J(V(p, 0)) = T(p, 0) \times \mathbf{C}^p/\mathbf{Z}^{2p}.$$

The projection of $T(p, 0) \times \mathbf{C}^p$ onto the first factor induces a holomorphic projection

$$\rho\colon J(V(p, 0)) \to T(p, 0),$$

such that for each $t \in T(p, 0)$, $\rho^{-1}(t)$ is the Jacobian variety $J(X^t)$ of the surface X^t (topologically, $J(X^t) = \mathbf{C}^p/\mathbf{Z}^{2p}$).

7.3. For fixed $t \in T(p, 0)$ and $\zeta_0 \in U(t)$, we define an *embedding* η of X^t into $J(X^t)$ by the formula

$$\eta_j(\zeta) = \int_{\zeta_0}^{\zeta} \alpha_j(t, \tilde{\zeta})\, d\tilde{\zeta}, \qquad 1 \leqslant j \leqslant p,$$

where $\eta = (\eta_1, \ldots, \eta_p)$ is viewed as a column vector. Formulae (7.1.4) and (7.1.5) show that equivalent points under Γ^t are mapped to the same point in $J(X^t)$.

7.4. We want to define an embedding of $V(p, 0)$ into $J(V(p, 0))$ whose restriction to each fiber X^t, $t \in T(p, 0)$, agrees with one of the maps given in §7.3. If we could choose the base point $\zeta_0 \in U(t)$ to depend holomorphically on t, we would be done. However, for $p > 2$ this is impossible by Theorem

6.2, so one has to proceed in an indirect manner (Earle [E5]). Define a map $\eta\colon F(\Gamma) \to \mathbf{C}^p$ by

$$(1-p)\eta_j(t,\zeta)$$
$$= -\frac{1}{2}\tau_{jj}(t) + \sum_{k=1}^{p} \int_{s=\zeta}^{A_k^t(\zeta)} ds \int_{u=\zeta}^{s} \alpha_j(t,u)\alpha_k(t,s)\,du, \qquad 1 \leqslant j \leqslant p.$$

Then, by direct calculation, η is holomorphic, $\partial\eta_j/\partial\zeta = \alpha_j$, $1 \leqslant j \leqslant p$,

$$\eta(A_k(t,\zeta)) = \eta(t,\zeta) + e_k,$$
$$\eta(B_k(t,\zeta)) = \eta(t,\zeta) + \tau(t)e_k, \qquad 1 \leqslant k \leqslant p,$$

where e_k is the kth-column of the identity matrix I.

THEOREM (EARLE [E5]). *The map $(t,\zeta) \mapsto (t, \eta(t,\zeta))$ from $F(\Gamma)$ to $T(p,0) \times \mathbf{C}^p$ defines a holomorphic embedding φ: $V(p,0) \to J(V(p,0))$ that sends each Riemann surface into its Jacobian variety.*

For more properties of this interesting map φ, see Earle [E5]. We shall mention only one remarkable fact.

7.5. The Jacobian variety of X^t, $J(X^t)$, can be identified with the divisor classes of degree zero on X^t (that is, divisors of degree zero modulo principal divisors). See, for example, Gunning [Gu, pp. 37–38]. The classical embedding of X^t into $J(X^t)$ is, in this setting, a holomorphic map that sends $x \in X^t$ to the class $[x - x_0]$ of the divisor $x - x_0$, where x_0 is the base point of the embedding. More generally, if $[D]$ is any divisor class of degree one on X^t, then $x \mapsto [x - D]$ is an embedding of X^t into $J(X^t)$. All translates of the classical embedding arise in this way. Hence, the embedding φ: $V(p,0) \to J(V(p,0))$ defines a divisor class $[D^t]$ of degree one on each X^t. Note that although it is not possible to find an integral divisor of degree one on each X^t which depends holomorphically on t, it is possible to find a class of degree one that does depend holomorphically on moduli. The class $[D^t]$ so found has the property that $[(p-1)D^t]$ is the divisor class corresponding to the vector of Riemann constants (see Fay [Fa, pp. 7–8] for definition). Hence, $[(2p-2)D^t]$ is the canonical class.

The canonical class $[K^t]$ depends holomorphically on t (Bers [B3]). Earle [E5] has shown that $x \mapsto [(2p-2)x - K^t]$ defines a holomorphical map of $V(p,0)$ into $J(V(p,0))$. Analytic continuation and the simple connectivity of $T(p,0)$ can be used to obtain a holomorphic embedding of the form $x \mapsto [x - K^t/(2p-2)]$.

7.6. We end with the following interesting

PROBLEM. *Fix an integer $p \geqslant 2$. What is the lowest value of n so that one can choose on each surface X^t ($t \in T(p,0)$) an integral divisor D_n^t of degree n in such a way that D_n^t varies holomorphically with t?*

It is not too hard to prove (as a result of §6.2) that this number $n(p)$ satisfies $n(2) = 1$, and for $p > 2$, $1 < n(p) \leqslant p - 1$.

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