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DEFINABLE DEGREES AND AUTOMORPHISMS OF $\mathcal{D}$

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The basic notion of relative computability, i.e. of one function $\alpha: N \longrightarrow N$ being computable from another β, defines, in the obvious way, first an equivalence relation $\alpha \equiv \beta$ on functions and then a partial ordering $\leqslant$, called Turing reducibility, on the equivalence classes, called Turing degrees. The analysis of the structure $\mathcal{D}$ of these degrees has been a central topic in recursion theory beginning with the papers of Post **[1944]** and Kleene and Post **[1954]**. We will here deal with a number of global or second order questions about $\mathcal{D}$ of the sort first raised in Rogers **[1967a]** and since then reiterated by many others. In particular we will show that many degrees and relations on them are definable purely in terms of the ordering answering some questions from Simpson **[1977]**. Our results also impose severe limitations on possible automorphisms of $\mathcal{D}$. Indeed every sufficiently large degree is fixed under every automorphism. (This answers questions from Rogers **[1967a]**, Simpson **[1977a]** and others.) By applying our methods to the principal filters (or cones) of $\mathcal{D}$ we can also considerably improve the solution to the homogeneity problem of Rogers **[1967]** given in Shore **[1979]** and **[1981]**. These results are all derived by combining a strengthening of Harrington and Kechris **[1975]** with the results and methods of Nerode and Shore **[1979]** and **[1980]** and Shore **[1979]**, **[1981]**. As in these latter papers a crucial role is played by the results of Jockusch and Soare **[1970]** on minimal covers and Lachlan **[1968]** on initial segments of $\mathcal{D}$.

LEMMA 1. *If the Turing degree* $\mathbf{x}$ *is not hyperarithmetic and* $\mathbf{x}, \mathbf{0}'' \leqslant \mathbf{t}$ *then there is a degree* $\mathbf{s}$ *such that* $\mathbf{t}$ *is a minimal cover of* $\mathbf{s}$ *and* $\mathbf{x} \nleqslant \mathbf{s}$.

PROOF. Our starting point is the proof of Harrington and Kechris **[1975]** that every Π_1^0 set of reals A such that every hyperarithmetic real is recursive in some member of A contains reals of every Turing degree above $\mathbf{0}''$. Let $A = \{\langle\sigma_0, \sigma_1, \tau_0, \tau_1\rangle | \sigma_1$ is a witness to the fact that every partial Π_1^1 function has a total extension recursive in σ_0 and τ_1 is the Skolem function witnessing that τ_0

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is the minimal cover of $\sigma = \langle \sigma_0, \sigma_1 \rangle$ given by the construction of Sacks [1963]}. As the desired property for σ_0 is Σ_1^1 and Sacks' construction gives a unique τ_0 which is Δ_2^0 in σ, A is a Π_1^0 set of reals. Consider now the game Harrington and Kechris associate with this set. We let I play initial segments of the various reals he is constructing but require that at each move the part of $\tau = \langle \tau_0, \tau_1 \rangle$ played so far can be seen to be on the appropriate tree recursive in σ using only the amount of σ played so far. This amount of σ should also suffice to verify the other conditions on I's play. We also require II to play a characteristic function.

Suppose σ results from a play of this game in which I follows the winning strategy of never moving to a position at which II has a winning strategy (a Σ_1^1 set of nodes). His play also produces τ, a minimal cover of σ. We claim that if $\{e\}^\sigma$ is total then at infinitely many positions p, I can, for each k, move to position q so as to force $\{e\}^{\sigma[q]}(k)$ to converge. ($\sigma[q]$ is the initial segment of σ determined by position q.) If not, we may as well assume that there are no such positions p. The relation that says that no winning move from p can force convergence at k is Π_1^1. It can therefore be uniformized by a Π_1^1 function and so there is a function f recursive in σ extending the uniformization. By our definition of A our play produces a $\tau = \langle \tau_0, \tau_1 \rangle$ which is on a tree recursive in the σ produced. We can now use this function f to prune the tree down to a finitely branching one. The point is that at any node p with I to move we calculate $f(p)$ and then find the amount n of σ needed to compute $\{e\}^\sigma(f(p))$. I can then only play initial segments of σ of length $< n$. This puts a finite bound on the possible number of extensions he can make to τ at this move. As II can play only a 0 or 1 at his move we have a class which is Π_1^0 in σ and recursively bounded in σ consisting precisely of the actual play of the game producing σ. Thus the entire play and so τ would be recursive in σ. This is the contradiction that establishes our claim.

Consider now any $\mathbf{t} \geqslant \mathcal{O}$. We let II play a characteristic function in $\mathbf{t}$. I plays to win of course but in addition at each move from any position p he asks for each $e < p$ not yet taken care of if he can find, for every k, winning moves which force $\{e\}^\sigma(k)$ to converge. This question can be answered for each $e < p$ recursively in $\mathcal{O}$ and so in $\mathbf{t}$. If there is such an $e <$ lth p he chooses the least one and takes care of it as follows: If there are two possible moves giving different values to $\{e\}^\sigma(k)$ for some k then he chooses one that makes $\{e\}^\sigma(k) \neq X(k)$. If not then $\{e\}^\sigma$ must be hyperarithmetic and so again is not in $\mathbf{x}$. Following this procedure I produces reals including σ and τ with $\tau \in \mathbf{t}$ a minimal cover of σ. Moreover if $\{e\}^\sigma$ is total then by our claim we could take care of it at infinitely many moves during the game. Thus after we have finished with all $e' < e$ for which we ever act we eventually take case of e and so guarantee that $x \nleqslant \sigma$.

Lemma 1 together with Jockusch and Soare [1970] enables us to define a jump ideal caught between those of the arithmetic degrees $\mathcal{A}$ and the hyperarithmetic ones $\mathcal{H}$.

LEMMA 2. $\mathcal{C} = \{x | \exists y [x \leqslant y \ \& \ \forall z (z \vee y$ *is not a minimal cover of* $z)]\}$ *is a jump ideal with* $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{H}$.

The methods of §2 of Nerode and Shore **[1980]** can now be used to prove the following theorems.

THEOREM 3. *Every degree and relation on degrees above all the hyperarithmetic degrees is definable in* $\mathcal{D}$ *iff it is definable in second order arithmetic.*

THEOREM 4. *Relations invariant under joining with an arbitrary hyperarithmetic degree such as "hyperarithmetic in", "hyperjump of" and "constructible in" are definable in* $\mathcal{D}$.

The relativized versions of the lemmas also show that if ϕ is an isomorphism of cones $\mathcal{D}(\geqslant \mathbf{a}) \longrightarrow \mathcal{D}(\geqslant \mathbf{b})$ then the image of every degree arithmetic in $\mathbf{a}$ is hyperarithmetic in $\mathbf{b}$. Combining this with initial segment results as in Nerode and Shore **[1980**, §4**]** or Shore **[1979]** we get the following results.

THEOREM 5. *If* $\mathcal{D}(\geqslant \mathbf{a}) \cong \mathcal{D}(\geqslant \mathbf{b})$ *then* $\mathbf{a} \equiv_h \mathbf{b}$.

THEOREM 6. *Every automorphism of* $\mathcal{D}$ *is the identity on every degree above all the hyperarithmetic ones.*

Finally using the coding methods of Nerode and Shore **[1980]** and Shore **[1981]** we can go from isomorphism results to ones on elementary equivalence.

THEOREM 7. *If* $\mathcal{D}(\geqslant \mathbf{a}) \equiv \mathcal{D}$ *then* $\mathbf{a}$ *is hyperarithmetic.*

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