HOMOLOGY OF GROUP SYSTEMS WITH APPLICATIONS TO LOW-DIMENSIONAL TOPOLOGY

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Eilenberg-Mac Lane complexes are generalized to *GEM complexes*. This generalization is then shown to unify many diverse seemingly unrelated concepts in low-dimensional topology. All 2-dimensional CW-complexes [1], all 3-dimensional manifolds [5], and all smooth 2-knot exteriors [5] are shown to be GEM complexes. A method is given for computing the (co)homology of the universal cover of a GEM complex from the (co)homology of a naturally associated group system. Hence, this yields a method for computing the second homotopy group π_2 and the k-invariant in $H^3(\pi_1; \pi_2)$.

I. GEM complexes.

DEFINITION. A *generalized Eilenberg-Mac Lane (GEM) complex* is a CW-complex K together with nonempty subcomplexes K_-, K_0, K_+ such that (1) $K = K_- \cup K_+$, (2) $K_0 = K_- \cap K_+$, (3) each K_λ is 0-connected and aspherical (i.e., $\pi_q K_\lambda = 0$ for $q \neq 1$) for $\lambda = -, 0, +$. The *associated group system* $\mathbf{G} = \pi_1 \mathbf{K}$ is the collection of groups, $\{\pi_1 K_-, \pi_1 K_0, \pi_1 K_+\}$ together with the morphisms induced by inclusion.

THEOREM 1. *Let K and K' be two GEM complexes. If an associated group system $\pi_1 \mathbf{K}$ of K is isomorphic to an associated group system $\pi_1 \mathbf{K}'$ of K', then K and K' are of the same homotopy type. Hence, the name "GEM" and the notation $K = K(\mathbf{G}, 1)$ are justified.*

THEOREM 2. *For every group system $\mathbf{G}$, the GEM complex $K(\mathbf{G}, 1)$ exists.*

REMARK 1. The exterior of every smooth 2-knot (S^4, kS^2) is a GEM complex since every 2-knot has a hyperbolic splitting. (See [5].) (This is a natural 4-dimensional analogue of the asphericity of classical knots [9].) Every 3-manifold is a GEM complex since every such 3-manifold has a Heegaard splitting of positive genus. Every 2-dimensional CW-complex has a subdivision which is a GEM complex [1].

II. Group systems.

DEFINITION. Let $\mathbf{G}$ be a group system and let G denote its direct limit

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(i.e., push-out). (See [3].) Let C^λ be a free left ZG_λ-resolution of the group G_λ for $\lambda = -, 0, +$. By a well-known theorem there is a chain map extension $\gamma^\pm \colon C^0 \longrightarrow C^\pm$ of the identity map of Z as a trivial left ZG_0 module onto Z as a trivial left $ZG_\pm$-module over the morphism $G_0 \longrightarrow G_\pm$. Let $\hat{C}^\pm$ denote the left $ZG_\pm$-chain complex which is the mapping cylinder of $\gamma^\pm$ and let $i_\pm \colon C^0 \longrightarrow \hat{C}^\pm$ denote the chain map induced by the inclusion of $ZG_\pm \otimes_{G_0} C^0$ into $\hat{C}^\pm$. Then the left ZG-chain complex which the direct limit (i.e., push-out) of the system $\hat{C}^- \xleftarrow{i_-} C^0 \xrightarrow{i_+} \hat{C}^+$ is called a *chain complex of the system* **G** and denoted by $C\mathbf{G}$.

PROPOSITION 3. *Every two chain complexes of a group system* **G** *are chain homotopic.*

DEFINITION. Let **G** be a group system with direct limit (i.e., push-out) G and let A be a right ZG-module. The *homology of the group system* **G** *with local coefficients in* A, written $H_*(\mathbf{G}; A)$, is defined as $H_*(\mathbf{G}; A) = H_*(A \otimes_G C\mathbf{G})$.

III. Main theorem.

THEOREM (MAIN) 4. *Let* **G** *be a group system with direct limit (i.e., push-out)* G. *Let* K *be the GEM complex* $K(\mathbf{G}, 1)$ *and let* $\widetilde{K}$ *denote the universal cover of* K. *Then every chain complex* $C\mathbf{G}$ *of* **G** *is chain homotopic to the augmented chain complex* $C\widetilde{K}$ *of singular chains of* $\widetilde{K}$. *Hence,* $\overline{H}_*\widetilde{K}$ *is isomorphic to* $H_*(\mathbf{G}; ZG)$ *as a left* ZG-module, *where* $\overline{H}$ *denotes the reduced homology.*

COROLLARY 5. *By the Hurewicz theorem, the second homotopy group* $\pi_2 K$ *is isomorphic to* $H_2(\mathbf{G}; ZG)$ *as a left* ZG-module. *By the Van Kampen theorem* [3], $G = \pi_1 K$.

REMARK 2. The above corollary may be thought of as a π_2-generalization of Crowell's version of the Van Kampen theorem [3]. (See §V.1). There are other versions of generalized Van Kampen theorems. See for example [2].

COROLLARY 6. *Let* $\overline{C}$ *be a free left resolution of* G *agreeing with* $C\mathbf{G}$ *up to and including dimension 2 and having* $C\mathbf{G}$ *as a direct summand. Then the map* $\overline{C}_3 \xrightarrow{\partial_3} B_2(\overline{C}) = Z_2(C\mathbf{G}) \xrightarrow{\eta} H_2(C\mathbf{G})$ *is a representative of the k-invariant* $\mathbf{k}K$ *of* K *lying in* $H^3(\pi_1 K; \pi_2 K)$. (See [7].)

COROLLARY 7. *A GEM complex* $K(\mathbf{G}, 1)$ *is aspherical, i.e., an Eilenberg-Mac Lane space, if and only if* $H_*(\mathbf{G}; ZG) = 0$.

The following is a generalization to push-outs of Swan's [10] Mayer-Vietoris sequence (with local coefficients) for free products with amalgamation.

COROLLARY 8. *Let* **G** *be a group system with push-out* G *and let* A *be a right* ZG-module. *Then the following is a long exact sequence*

$$\cdots \longrightarrow H_{q+1}(\mathbf{G}; A) \longrightarrow H_q(G_0; A)$$

$$\longrightarrow H_q(G_-; A) \oplus H_q(G_+; A) \longrightarrow H_q(\mathbf{G}; A) \longrightarrow \cdots.$$

IV. **Methods for computing** $H_*(G; ZG)$. To each presentation $(x:r)$ of a group G there corresponds a Fox-Lyndon resolution where the boundary operators are given by the Jacobians of the Fox derivatives. (See [4, II, p. 210], [6, p. 651], and [11, p. 471].) In [5] a method is given for computing a presentation of the left ZG-module $H_*(G; ZG)$ from the Fox-Lyndon resolutions of the groups in G. The kernels of the boundary operators are computed in [5] using the techniques of [12, §8]. These methods were used in [5] to compute the algebraic 3-type [7] of smooth 2-knot complements and of 3-manifolds.

V. **Generalizations.** Details for the following will appear elsewhere.

1. The phrases "aspherical structure" and "group system" are used above rather than "aspherical triad" and "group triad" because all definitions and results listed above hold for more general aspherical structures and group systems.

2. All the definitions and results of this paper may be generalized to GEM complexes $K(G, n)$ of type n.

3. The cohomology of the universal cover of GEM complexes and of group systems can also be treated as above.

VI. **Sketch of proofs.** (1) Theorem 1 is proven by piecing together compatible homotopy equivalences between corresponding sets in the aspherical structures. The existence of such compatible equivalences follows from standard arguments in obstruction theory. The resulting map is shown to be a homotopy equivalence again by standard arguments in obstruction theory. (2) Theorem 2 follows by first applying Milnor's classifying space functor [8] to the group system and then making use of the appropriate reduced mapping cylinders. (3) Main Theorem: Cellular chain complexes C^-, C^0, C^+ of the universal covers of K_-, K_0, K_+ are formed by lifting cell decompositions of K_-, K_0, K_+. An appropriate chain map $\gamma^\pm\colon C^0 \longrightarrow C^\pm$ over $\pi_1 K_0 \longrightarrow \pi_1 K$ is chosen. The push-out is then shown to be exactly the cellular chain complex $C\widetilde{K}$ of the universal cover $\widetilde{K}$ of K obtained by lifting the cell decomposition of K to $\widetilde{K}$.

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