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Characterizations of probability distributions, by Janos Galambos and Samuel Kotz, Lecture Notes in Math., vol. 675, Springer-Verlag, Berlin-Heidelberg-New York, 1978, viii + 169 pp., $9.80.

Characterization of functions by their properties is a mathematical pursuit of long standing. If one is interested in phenomena in which the observable quantities are subject to chance variation, the most important "functions" are probability distributions which describe the chance variation. The problem of characterization of probability distributions can be described, in general terms, as follows: It is known that a family of distributions $\mathfrak{F}$ possesses a certain property $\mathfrak{P}$; is it true, conversely, that a distribution has the property $\mathfrak{P}$ only if it is a member of $\mathfrak{F}$? If so, $\mathfrak{P}$ characterizes the family $\mathfrak{F}$. This result is then referred to as a "characterization of the ($\mathfrak{F}$) distribution", in keeping with the universal practice of using the term "distribution" interchangeably to mean either a single distribution or a family of distributions labelled by the same name. Although characterization problems can be of intrinsic mathematical interest, a principal motivation for this research has been provided by practical considerations: If a statistical procedure is valid only when property $\mathcal{P}$ holds, and this property characterizes the family $\mathcal{F}$, then the procedure is valid only in those situations in which the underlying distribution is a member of the family $\mathcal{F}$.

As in many other areas of mathematics, research in the characterization of distributions started initially with a few results which made their appearance sporadically; later the tempo of activity increased, resulting in a seemingly self-reproducing aggregate which now consist of results of varying quality: a few are mathematically simple but are of far-reaching significance in theory and practice, there are some whose proofs require extremely hard work and substantial mathematical prowess but might have no practical significance, and there are others which are mathematically rather trivial and are also of doubtful significance. A scholar interested in this field requires a capable and discriminating guide to point out to him (or her) the beautiful flowers or useful herbs and by-pass the weeds. This review can only attempt a brief survey.

The earliest paper devoted to characterization of a distribution was written by George Pólya [9], and proved the following result: If X, Y are independent random variables having the same distribution $\mathcal{F}$ which has a finite second moment, and if there exist positive constants a, b such that $aX + bY$ also has the same distribution F, then F is a normal distribution with zero mean; i.e., F is absolutely continuous with a density function f of the form $f(x) = (a/\sqrt{2\pi})\exp(-a^2x^2/2)$ for some $a > 0$. The next characterization result [2] dealt with another property of the normal distribution which had been found useful for statistical inference: If $X_1, \ldots, X_n$ are independent random variables with the same distribution F, then $\bar{X} = (X_1 + \cdots + X_n)/n$ and $S = \Sigma(X_i - \bar{X})^2$ are independent random variables if F is a normal distribution. Geary showed that if $\bar{X}$ and S are independent and F has a moment generating function, then F is normal. A third interesting property of the normal distribution, which has been of considerable use in statistical inference, was used for characterization by Kac [3] and Bernstein [1] who proved, independently of each other, that if X, Y are independent random variables, then $X + Y$ and $X - Y$ are independent random variables only if X, Y are normally distributed. Macinkiewicz [8] picked up the theme of Pólya's paper and proved the following result: If $\{X_n\}$ is a sequence of independent random variables all having the same distribution F (which has finite moments of all positive orders) and there exist two linear forms $\Sigma a_n X_n$ and $\Sigma b_n X_n$ whose distributions are identical, then F is normal. Lukacs [7] removed some of the restrictive assumptions of Geary's paper. Pólya's paper appeared in 1923, and the other five during the period 1936–1942; the next contribution in the field appeared in 1949. The tempo of activity has steadily increased since then. The early papers of the post-war era represent a quantum jump over the pre-war achievements. Kawata and Sakamoto [5] and Zinger [11] independently got rid of Geary's restrictive moment assumptions;

Skitovič [**10**] extended the Kac-Bernstein result to an arbitrary (but finite) number of random variables; and Linnik [**6**] solved the Pólya-Marcinkiewicz problem in substantial generality. Linnik's monumental paper still stands out as an impressive achievement.

The work described in the previous paragraph makes substantial use of the theory of characteristic functions (i.e., Fourier transforms) of probability distributions, developed by Paul Lévy in the 1920's, which had been extensively exploited in other areas of probability theory. Characteristic functions continue to be useful in characterization problems, but are not a universal panacea. The nature of the property and the characterized distribution determine the mathematical tools appropriate to the problem; there does not exist a unified theory of characterization of distributions. Most of the properties investigated are of the type described in the previous paragraph, in the sense that they are defined directly in terms of distributions of functions of random variables; but some are defined in terms of concepts from the theory of statistical inference (such as sufficient statistics, optimal estimators, etc.). The number of different distributions which constitute the main-stay of statistical theory is rather small. Although most of them have now appeared in the literature on characterization, the normal distribution gets the lion's share of attention. To some extent, this is due to the early predominance of the normal distribution in statistical theory; but the number of convenient analytical properties which the normal distribution possesses might have played a role. Therefore, the following question seems pertinent to any attempt at assessing the future of the subject: What opportunities do other distributions provide for characterization? One way of finding an answer is to look at what has been done for other distributions. The exponential distribution is one nonnormal distribution which has received considerable attention in recent years.

The monograph under review restricts itself primarily to the exponential distribution (univariate and multivariate) and its monotone transformations. By the univariate exponential distribution is meant the two-parameter family of probability distributions on the real line corresponding to the family of density functions $\{f(\cdot; a, b), a > 0, b \in R^1\}$, where $f(x; a, b) = a \exp[a(b - x)]$ for $x > b$ (and $f = 0$ for $x < b$). Experience has shown this distribution to be appropriate to a wide range of seemingly disparate phenomena (such as, for example, emission of particles from a radio-active source and arrivals of calls at a telephone). The monograph covers its chosen ground in detail. There are some misprints and an occasional error.

Much of the work on the exponential distribution is mathematically simple. By contrast, the normal distribution has provided some challenging problems. The book by Kagan, Linnik and Rao [**4**] is a valuable source of information about the whole field of characterization of probability distributions. It gives the reader a good idea of the different types of properties which have been used for characterization, and of the mathematical intricacy of the problems. Looking over the developments of recent years, one gets the impression that although the frequency of contributions has increased, that of substantial results achieved a maximum earlier. Perhaps there is a need for new high-yield properties. No method of discovering such properties is known; the field is therefore open to wild-cat explorers. Does it contain untapped reserves rich enough to ensure major strikes? The answer, by hindsight, may perhaps be found in these pages some years hence.

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