of the book. Applications are drawn primarily from many-valued logic, an area traditionally (though perhaps unjustly) linked to philosophy. Trains of thought that would naturally occur to a mathematician (or, at least, did occur to me) are omitted entirely. For example, there is no mention of the fact that, in discussing graph arguments from X to Y, one loses no generality by assuming that X and Y are both empty. ($X \vdash Y$ follows from a set of inferences $X_i \vdash Y_i$ if and only if $\varnothing \vdash \varnothing$ follows from these inferences along with $\varnothing \vdash A$ (resp. $A \vdash \varnothing$) for every formula A in X (resp. Y); in other words, ∇ and Δ can be replaced with $\overset{\top}{\circ}$ and $\underset{\perp}{\circ}$ respectively.) And, finally, the detailed motivations of concepts and theorems go well beyond the norm to which mathematicians are (unfortunately) accustomed.

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Invariant theory, by T. A. Springer, Lecture Notes in Math., vol. 585, Springer-Verlag, Berlin, Heidelberg, New York, 1977, 111 pp., $8.00.

Invariant theory. The very words recall potent historical forces. Hilbert **[10]** viewed mathematical theory as the sum of three stages of development: the naive, the formal, and the critical periods. The progenitors of these periods come to mind. The Naive Period of invariant theory is represented by Boole, Sylvester, and Cayley, those conjurers of catalectants and other invariants of special quantics. The Formal Period arrived with the work of the Italian school of Cremona, Beltrami, and Capelli and the German school of Aronhold, Clebsch, and Gordon, whose symbolic method exposed the power of duality in algebra. In the Critical Period the heretic Hilbert reigned, armed with his homological methods; ultimately Noether, Van der Waerden and Artin enlarged on his ideas to found modern algebra.

Although Hilbert declared the subject dead in 1893 **[9]**, rumors of its demise were greatly exaggerated. Soon after, Reverend Young, alone and unnoticed, was divining the secrets of the symmetric group from his diagrams. At the same time, Weitzenböck, Study, and Littlewood unmasked tensor analysis as invariant theory in disguise. Soon after Molien, Frobenius, Cartan, Schur, and Weyl in generalizing invariant theory, ensconced it within a new subject, representation theory. No wonder Dieudonné could jest, "Invariant Theory has already been pronounced dead several times and like the phoenix it has been again and again arising from its ashes," **[3]**.

Has it been laid to rest? Hardly! The recent International Congress of Mathematicians in Helsinki included at least three forty-five minutes addresses devoted largely to recent progress in the field. Invariant theory is like the roots of a great tree, whose branches touch all of mathematics; still in its prime, it is bearing beautiful fruit. Consider some applications of invariant theory: in geometry, the study of the conformal invariance of characteristic classes by Chern [4] and Simons [19]; in algebra, the work of Haboush [8] and Procesi [16] on the Mumford conjecture; in analysis the work of Gilkey [7] and Patodi's [15] study of invariants of the heat equation; and in combinatorics, Rota's work on Cayley spaces [18] and Stanley's on magic labeling of graphs [21] and the upper bound conjecture [22].

How this subject's popularity can swing so widely in only seven score years is a fascinating question in its own right, and I recommend the *Death of a mathematical theory* by Fisher [5] for an analysis of the question.

Having sketched the historical position of invariant theory,[1] I want to illustrate some of the problems and methods of the subject. The historical periods outlined above provide a natural setting for several apt examples.

The naive period. Invariant theory is agreed to have begun with Boole's paper [1] in 1841, which inspired Cayley, Sylvester and the British school on Invariant Theory. They relied on ad hoc geometric techniques to confront such problems as, for example, finding all invariants of a pair of linear forms over the reals

$$ax + by \quad \text{and} \quad cx + dy$$

under the group of rotations in the (x, y)-plane. What this really means is the following. A rotation through the angle θ takes a point (x^*, y^*) into (x, y) via the formula

$$x = x^*\cos\theta - y^*\sin\theta \quad \text{and} \quad y = x^*\sin\theta + y^*\cos\theta,$$

and substitution of these formulas into the linear forms yields

$$\begin{aligned} ax + by &= (a\cos\theta + b\sin\theta)x^* + (b\cos\theta - a\sin\theta)y^* \\ cx + dy &= (c\cos\theta + d\sin\theta)x^* + (d\cos\theta - c\sin\theta)y^*, \end{aligned}$$

which induces an action on (a, b, c, d):

$$\begin{aligned} a^* &= a\cos\theta + b\sin\theta \quad & b^* &= b\cos\theta - a\sin\theta \\ c^* &= c\cos\theta + d\sin\theta \quad & d^* &= d\cos\theta - c\sin\theta. \end{aligned}$$

The problem then is to find all polynomials in four variables over the integers that satisfy

$$P(a, b, c, d) = P(a^*, b^*, c^*, d^*).$$

A direct geometric analysis shows that the only geometric invariants are the length of the forms and the angle between them. The only polynomials arising from these quantities are of even degree and are generated as an algebra by

$$a^2 + b^2, c^2 + d^2, ac + bd, ad - bc.$$

These generators satisfy the relation

$$(ac + bd)^2 + (ad - bc)^2 = (a^2 + b^2)(c^2 + d^2)$$

which is classically called a syzygy, and the algebra of invariant polynomials turns out to be the homomorphic image of the polynomial algebra in four

[1]The definitive history of invariant theory up to 1891 is Meyer's report to the Deutsche Mathematische Verein [13].

variables given by the quotient algebra

$$Z[x, y, z, w] / (z^2 + w^2 - xy).$$

In particular, the algebra is finitely generated by four explicit polynomials, and the ideal of relations is finitely generated by a single explicit relation.

The successes of this naive period were mainly of the type just illustrated; that is, a geometric configuration (such as a pair of quadratic forms in two variables, or a quartic in two variables) was studied for invariants under some group of linear transformations. The analysis generally involved difficult computation and geometric ingenuity; nevertheless the status of the young subject rose rapidly. As Weyl **[24]** (2), looking back, eulogized: "The Theory of Invariants came into existence about the middle of the nineteenth century somewhat like Minerva: a grown-up virgin, mailed in the shining armor of algebra, she sprang forth from Cayley's Jovian head. Her Athens over which she ruled and which she served as a tutelary and beneficent goddess was projective geometry."

The formal period. This stage was marked by two major achievements: the structure of vector relative invariants, and the development of the symbolic method. The first, due largely to Cayley's invention of the Ω process and the establishment of the Capelli identity, is known as the first fundamental theorem of vector relative invariants for the general linear group. The ideas basic to this result can be quickly reviewed.

Let K be a field of characteristic zero, V an n-dimensional vector space, and W and m-dimensional vector space, both over K. If

$$\rho\colon \mathrm{Aut}_K(V, V) \to \mathrm{Aut}_K(W, W)$$

is a homogeneous polynomial representation, then a homogeneous polynomial mapping

$$f\colon W \to K$$

is called a *relative invariant* under ρ, if there exists a character

$$\chi\colon \mathrm{Aut}_K(V, V) \to K$$

such that for all $g \in \mathrm{Aut}_k(V, V)$ and for all $x \in W$

$$f(\rho(g)x) = \chi(g)f(x).$$

Since every such character χ is a power of $\det g$, this implies that

$$f(\rho(g)x) = (\det g)^k f(x),$$

where k is called the weight. Viewing this last equation as a polynomial map of n^2 variables for g and n variables for x, one has

$$\deg f \deg \rho = k \dim_K V. \tag{$*$}$$

The first fundamental theorem of relative vector invariants is concerned with the case

$$W = \otimes^p V \text{ and } \rho = \otimes^p.$$

It states that if a linear mapping

$$f\colon \otimes^p V \to K$$

is a relative invariant, it is identically zero, unless p is divisible by n, in which case f is characterized on monomials as a linear combination of mappings of the form

$$f(x_1 \otimes \cdots \otimes x_{kn}) = \det(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \cdots \det(x_{\sigma((k-1)n+1)}, \ldots, x_{\sigma(kn)}),$$

where $\sigma \in S_p$, the permutation group on p letters. (See [6] for a modern proof.)

The second achievement of the formal period was the development of a general method which, albeit a computational nightmare, led to a procedure for reducing the calculation of relative invariants of mixed tensors to the calculation of relative invariants of vectors and covectors. Largely the work of Aronhold, Clebsch, Gordon, and Weitzenböck, this reduction is known as the symbolic method. To illustrate these ideas, I will restrict myself to relative invariants for tensors of type $(p, 0)$, reserving remarks on the generalization to mixed tensors till later.

The basic idea is to consider the image of an invariant under a duality isomorphism restricted to a particular spanning set. This image determines the invariant, and is in principle computable from the first fundamental theorem. Once the image is known, it is a formal process to recover the original invariant.

For instance, let W be an invariant subspace of $\otimes^p V$ under the representation $\otimes^p$; such a subspace is called a *symmetry class of* $(p, 0)$ *tensors*. A polynomial relative invariant f of degree d of elements of W is an element of $S^d(W)$, the polynomials of degree d defined on W. The first step used in the symbolic method is simply the duality isomorphism of $S^d(W)$ and $(\circledcirc\, {}^d W)^*$, the symmetric d-multilinear mappings on W. (Note that the field K must be infinite for this identification.) Although classical calculations needed an explicit description of the isomorphism

$$S^d(W) \simeq (\circledcirc\, {}^d W)^*.$$

It now suffices to only know the existence of the isomorphism which is easily established by universal mapping properties. The mapping

$$P\colon S^d(W) \to (\circledcirc\, {}^d W)^*$$

is classically known as polarization. Thus, given a polynomial relative invariant $f \in S^d(W)$, then $P(f)$ is a symmetric multilinear mapping

$$P(f)\colon\ \underbrace{Wx \ldots xW}_{d} \to K.$$

Since $W \subset \otimes^p V$, $Wx \ldots xW \subset \otimes^p Vx \ldots x \otimes^p V$ and $P(f)$ induces a linear relative invariant

$$\mathfrak{S}(f)\colon \otimes^{pd} V \to K,$$

which can now be described by the first fundamental theorem.

The second step in the symbolic method was to study these maps not on all elements of W, but only on a convenient spanning set. The usual example has $W = S^p V$, in which case it suffices to know the map on pure pth powers of elements of V, since the span of such elements gives all of W. (Note that the rationals $Q \subset K$ are necessary to verify this statement.) The restriction of $\mathfrak{S}(f)$ to the elements coming from pure pth powers of V is a determinantal expression involving d vectors; this is called *the symbolic expression* for f. The process of going back through these isomorphisms from the symbolic expression is classically called restitution.

Let us look at the simple example of finding the relative invariants of a binary quadratic

$$Q(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2.$$

In this case $V = K^2$ and $W = S^2V$. It is most convenient to parameterize $Q \in S^2V$ via the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

for then we want polynomials $f(A)$ that are relative invariants under the action

$$\rho(T)A = {}^tTAT.$$

Now comes one of the stumbling blocks of the symbolic methods: we must consider the invariants one degree at a time.

Thus we begin with relative invariants f of degree 2, which are elements of $S^2(S^2V)$. Polarization gives rise to a bilinear map

$$P(f)\colon S^2V \times S^2V \to K,$$

which produces a linear mapping

$$\mathfrak{S}(f)\colon \otimes^4 V \to K.$$

Next we must restrict this to pure powers in S^2V. The pure powers in S^2V are the elements of the form

$$Q(x, y) = (v_1x + v_2y)^2 = v_1^2x^2 + 2v_1v_2xy + v_2^2y^2$$

and hence are those elements parameterized by matrices of the form

$$A = \begin{pmatrix} v_1^2 & v_1v_2 \\ v_1v_2 & v_2 \end{pmatrix}$$

or, equivalently, by elements in $S^2V \subset V \otimes V$ of the form

$$v \otimes v \quad \text{where} \quad v = {}^t(v_1, v_2).$$

Once the formula has been deduced for these special elements then we get the formula for the general elements by restitution which replaces

$$v_1^2 \text{ by } a_{11},\ v_1v_2 \text{ by } a_{12}, \quad \text{and} \quad v_2^2 \text{ by } a_{22}.$$

The restriction of $P(f)$ to pure power elements gives

$$P(f)\left(\begin{pmatrix} v_1^2 & v_1v_2 \\ v_1v_2 & v_2^2 \end{pmatrix}, \begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}\right) = \mathfrak{S}(f)(v \otimes v \otimes u \otimes u)$$

where $v = {}^t(v_1, v_2)$ and $u = {}^t(u_1, u_2)$. Among the possible terms allowed for $\mathfrak{S}(f)$ by the fundamental theorem, only one is nontrivial. Thus, all the invariants have a symbolic expression

$$\mathfrak{S}(f)(v \otimes v \otimes u \otimes u) = c \det(v, u)^2 \quad \text{with } c \text{ a constant.}$$

Since

$$\det(v, u)^2 = v_1^2u_2^2 + u_1^2v_2^2 - 2v_1v_2u_1u_2,$$

restitution gives

$$P(f)\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} = a_{11}b_{22} + b_{11}a_{22} - 2a_{12}b_{12}$$

and diagonal evaluation recovers

$$f\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = 2(a_{11}a_{22} - a_{12}^2).$$

This completes the consideration of degree 2 invariants. It is also possible to describe completely the invariants of all degrees for this special case of binary quadrics via the symbolic method (see Weyl [**25**, pp. 246–247]), but Weyl's ideas do not generalize to higher dimensions.

The most difficult process in the symbolic method is the restitution step. This was not visible in the last example, where it simply consisted of replacing the elements v_iv_j by the matrix elements a_{ij}. Since the algebra of the 19th century lacked the setting we were able to give for this procedure, the expositors of the time found something mysterious in the step. Most papers stressed that the v_i, which never appear alone, do not exist by themselves. Sylvester went so far as to coin the term "umbra" for these objects, and called the techniques of the symbolic method the umbral calculus.

Since the first example of the symbolic method was so simple, a sketch of a more complicated example may be helpful. Let us consider the problem of finding the relative invariants of binary quartics treated in Mumford [**14**]. Thus let

$$Q(x, y) = a_4x^4 + a_3x^3y + a_2x^2y^2 + a_1xy^3 + a_0y^4$$

and take

$$V = K^2 \quad \text{and} \quad W = S^4V.$$

The pure powers in S^4V look like

$$v^4 = (v_1x + v_2y)^4 = v_1^4x^4 + 4v_1^3v_2x^3y + 6v_1^2v_2^2x^2y^2 + 4v_1v_2^3xy^3 + v_2^4y^4;$$

hence the restitution process replaces

$$v_1^{4-i}v_2^i \quad \text{by} \quad \binom{4}{i}^{-1} a_i.$$

A relative invariant of degree 2 is an element $f \in S^2(S^4(V))$, which gives rise to a linear relative invariant $\mathcal{S}(f)$: $\otimes^8 V \to K$, whose symbolic expression is

$$\mathcal{S}(f)\,(v \otimes v \otimes v \otimes v \otimes w \otimes w \otimes w \otimes w) = c \det(v, w)^4,$$

with c a constant. Since

$$\det(v, w)^4 = v_1^4w_2^4 - 4v_1^3v_2w_1w_2^3 + 6v_1^2v_2^2w_1^2w_2^2 - 4v_1v_2^3w_1^3w_2 + v_2^4w_1^4,$$

restitution produces $P(f)$ as

$$a_4b_0 - \tfrac{1}{4}a_3b_1 + \tfrac{1}{6}a_2b_2 - \tfrac{1}{4}a_1b_3 + a_0b_4$$

and diagonal evaluation yields f as

$$\tfrac{1}{6}(a_2^2 - 3a_1a_3 + 12a_0a_4).$$

Similarly, there is a relative invariant of degree three with symbolic expression

$$\mathfrak{S}(f)(v \otimes v \otimes v \otimes v \otimes w \otimes w \otimes w \otimes w \otimes u \otimes u \otimes u \otimes u)$$
$$= c \det(v, w)^2 \det(w, u)^2 \det(u, v)^2.$$

Restitution and diagonal evaluation produces f as

$$a_0a_2a_4 - \tfrac{3}{8}a_0a_3^2 - \tfrac{3}{8}a_1^2a_4 + \tfrac{1}{8}a_1a_2a_3 - \tfrac{1}{36}a_2^3.$$

The analogous problem for mixed tensors is handled by using the Poincaré duality of $\mathrm{Gl}(V)$-modules

$$\phi\colon V^* \to \Lambda^{n-1}V$$

to replace covectors by vectors. Note that this will result in a shift of weight for relative invariants, since given $S \in \mathrm{Gl}(V)$ and $\omega \in V^*$, then

$$\Lambda^{n-1}(S)\phi(\omega) = (\det S)\phi({}^tS^{-1}\omega).$$

The identification, ϕ, which yields the embedding

$$\otimes^p V \otimes \otimes^q V^* \to \otimes^{p+(n-1)q} V,$$

was the discovery of Weitzenböck and reduces the invariant theory of mixed tensors to the symbolic method. Note that the process of restitution is now more complicated, since the invariants have to be factored through this last identification.

The full import of the symbolic method remains somewhat unclear. In the early 1940's, Littlewood opined "The most effective work in classical invariant theory is based on the symbolic method" [**12**, p. 306], while only a few years earlier Weyl averred "Certainly the importance of the symbolic method, whose formal elegance nobody will deny, has been greatly exaggerated" [**24**, p. 499].

In any event even the lowest dimensional cases of the symbolic method are still finding modern applications. Rota and Roman [**17**] for example, have applied these methods to combinatorial problems. The connection between their umbral calculus and my description deserves a word of comment. There is a classical, weighted identification between binary n-ics and polynomials in one variable of degree n, given by

$$\sum \binom{n}{i} a_i x^{n-i} y^i \to \sum a_i y^i.$$

Hence pure powers on the left lead to

$$(v_1x + v_2y)^n = \sum \binom{n}{i} v_1^{n-i} v_2^i x^{n-i} y^i \to \sum v_2^i y^i$$

and restitution provides the substitutions

$$a_i = v_i^{n-1} v_2^i \rightarrow a_i = v_2^i.$$

The substitution on the right is the umbra discussed by Rota.

The critical period. By 1888 students of invariant theory had spent twenty years trying to apply the symbolic method to ternary homogeneous polynomials, in the hope of proving the analog of Gordon's 1868 finiteness theorem which showed that binary homogeneous polynomials have only a finite number of independent invariants. Hilbert's iconoclastic basis theorem, finally broke the log jam for it was the key to proving not only the above ternary case but also the general finiteness theorem for n-ary homogeneous polynomials. His work on invariant theory continued for five years, culminating in the 1893 paper that introduced homological dimension and proved his famous syzygy theorem.

Let us outline Hilbert's proof of the general finiteness theorem, he first showed that given any set $\mathfrak{S} \subset K[x_1 \ldots x_n]$, a polynomial ring over a field, there exists a finite number of polynomials $f_1, \ldots, f_n \in \mathfrak{S}$ such that each $f \in \mathfrak{S}$ can be expressed in the form

$$f = \Sigma\, a_i f_i \quad \text{with} \quad a_i \in K[x_1, \ldots, x_1].$$

This is applied to invariant theory by taking $K[x_1, \ldots, x_n]$ to be the polynomial ring in the coefficients of the n-ary homogeneous polynomials, and by letting the set $\mathfrak{S}$ be the set of relative invariants. Any invariant f can then be expressed in terms of a finite set of invariants $f_1, \ldots, f_n$ as

$$f = \Sigma\, a_i f_i \quad \text{with} \quad a_i \in K[x_1, \ldots, x_n].$$

To complete the finiteness theorem, one must show that the a_i can be chosen in $\mathfrak{S}$. This required Cayley's Ω process which is a projection operation

$$\Omega\colon K[x_1, \ldots, x_n] \rightarrow \mathfrak{S}.$$

Hilbert also was able to solve the second main problem of invariant theory which was concerned with how uniquely the $f_1, \ldots, f_n$ represented an invariant. One of the first questions addressed was whether the syzygies (i.e., the relations among the $f_1, \ldots, f_n$) were finitely generated. It was doubly astounding to the mathematicians of the time that not only the general finiteness theorem, but also the finiteness of the syzygies was an immediate consequence of the basis theorem simply by taking

$$\mathfrak{S} = \{g \in K[y_1, \ldots, y_n] \,|\, g(f_1, \ldots, f_n) = 0\},$$

which is an ideal B_1.

Since the second main problem had succumbed so easily, it was natural to turn to chains of syzygies, studying relations among the generating set of relations, and so on. More precisely, this work involved the sequence of finitely generated $K[y_1, \ldots, y_n]$-modules

$$
\begin{array}{ccccccccc}
0 & \to & J_2 & \to & F_1 & \to & B_1 & \to & 0 \\
0 & \to & J_3 & \to & F_2 & \to & J_2 & \to & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \to & J_q & \to & F_q & \to & J_{q-1} & \to & 0,
\end{array}
$$

where the F_i are free with rank equal to the minimal number of generators of the ith syzygies J_i. The most obvious questions are how unique are the J_i and the F_i, and what invariants can be deduced from them. Hilbert's main theorem on the chains of syzygies says that the $J_q = 0$ if $q > n$. In effect, this launched the theory of homological dimension of rings. A good review of Hilbert's various contributions during this period is Weyl **[23]**.

Post-Hilbert period. As mentioned in the introduction, although Hilbert declared invariant theory dead in 1893, not all believed his proclamation. One important advance that belied his pessimism was the extension of the methods of invariant theory to other groups especially the orthogonal group, the sympletic group and finite groups. Strictly speaking this activity belongs to the Critical Period, but it has such far ranging applications as a calculational tool that I choose to label it otherwise. In celebrating invariant theory as the ultimate technique of tensor analysis Weitzenböck and Littlewood were absolutely correct. Let me illustrate this by one example of an application of the first fundamental theorem of orthogonal vector invariants, which states that any polynomial invariant of n vectors for the orthogonal group is a polynomial in determinants and scalar products.

The example is the computation of an integral that arises in the physics of a radiation filled universe:

$$\int_{S^3} (x \cdot a)(x \cdot b)(x \cdot c)(x \cdot d)\, dA$$

where $a, b, c, d \in \mathbf{R}^4$ and x is the position vector of the 3-sphere. The integral is a function $f(a, b, c, d)$ which is invariant under the orthogonal group. This follows because a simultaneous orthogonal transformation on a, b, c, d can be pushed off onto x and absorbed by the orthogonal invariance of the measure on S^3. Since $f(a, b, c, d)$ is linear in a, b, c, d and symmetric in all entries, the first fundamental theorem of orthogonal invariants implies

$$f(a, b, c, d) = \lambda[(a \cdot b)(c \cdot d) + (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$$

where λ has yet to be determined. Now, this formula must hold for all a, b, c, d hence for $a = b = c = d$ and $|a| = 1$, which yields

$$\int_{S^3} (x \cdot a)^4\, dA = \lambda.$$

If we take Y to be the position vector of the equatorial 2-sphere with pole a, we have

$$x = \sin \phi Y + \cos \phi a$$

and

$$\int_{S^3} (x \cdot a)^4 \, dA = \int_0^{\pi} (\cos \phi)^4 (\sin \phi)^2 \, d\phi \int_{S^2} dA$$
$$= \frac{1}{32\pi} 4\pi$$

from which we deduce $\lambda = \frac{1}{8}$. A similar but considerably more sophisticated calculation appears in Weyl's paper, *On the volume of tubes* **[26]**.

In a different vein, considerable recent work has focused on Hilbert's fourteenth problem, which asks what groups satisfy the finiteness theorem. The famous 1958 counterexample of Nagata [3] showed that the finiteness theorem is not always true, and Mumford's conjecture that reductive algebraic groups are geometrically reductive has been proven by Haboush **[8]**. J. Humphreys has recently written an elementary survey on these questions **[11]**.

Attention has also been given to explicit computations of rings of invariants for given groups and to identifying the groups whose rings of invariants are polynomial rings. The most satisfactory results are for finite groups, where the finiteness theorem is known to hold. Molien developed an explicit representation for the power series whose ith coefficient is the dimension of the vector space of invariant polynomials of degree i. I recommend to the reader Sloane's **[20]** for interesting applications of this theorem to the computation of invariants, and their applications to coding theory. Which finite groups have polynomial rings for rings of invariants has been answered by Chevalley [2]; they are the generalized reflection groups.

The review. Having gotten the bird's eye view of the field, we can now turn to Springer's book. By his own admission, these notes give a very incomplete picture of invariant theory. The subject matter derives primarily from the Critical and Post-Hilbert periods, although many of Springer's examples were understood earlier. The notes are an enjoyable, readable account of the invariant theory of reductive algebraic groups, concentrating on delicate finiteness theorems. The general theory is illustrated by a detailed analysis of $\mathrm{Sl}(2, K)$ and finite groups. In particular, the above mentioned theorems of Molien and Chevalley-Serre are clearly presented and lead to interesting explicit calculations for classical reflection groups.

I especially recommend these notes to any mathematician who wonders why finiteness theorems are important and how concepts like the integrality of extensions and Noether normalization arose historically. The author has included many references and notes at the end of each chapter, indicating where various results first occurred and why they're significant.

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Characterizations of probability distributions, by Janos Galambos and Samuel Kotz, Lecture Notes in Math., vol. 675, Springer-Verlag, Berlin-Heidelberg-New York, 1978, viii + 169 pp., $9.80.

Characterization of functions by their properties is a mathematical pursuit of long standing. If one is interested in phenomena in which the observable quantities are subject to chance variation, the most important "functions" are probability distributions which describe the chance variation. The problem of characterization of probability distributions can be described, in general terms, as follows: It is known that a family of distributions $\mathfrak{F}$ possesses a certain property $\mathfrak{P}$; is it true, conversely, that a distribution has the property $\mathfrak{P}$ only if it is a member of $\mathfrak{F}$? If so, $\mathfrak{P}$ characterizes the family $\mathfrak{F}$. This result is then referred to as a "characterization of the ($\mathfrak{F}$) distribution", in keeping