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Differenzenapproximationen partieller Anfangswertaufgaben, by R. Ansorge, Leitfäden der angewandten Mathematik und Mechanik (LAMM), Volume 45, B. G. Teubner, Stuttgart, 1978, 298 pp., DM 29.80.

Exactly thirty years ago, when I was about to develop a serious interest in some numerical aspects of partial differential equations, two well-known mathematicians gave me the benefit of their deeper insights in the form of two predictions. "Digital computing machines will never successfully compete with analog computers. Their vaunted speed is no use, since they break down all the time" was one prediction. "The role of functional analysis in the theory of partial differential equations will always remain mostly decorative. The important ideas can equally well be expressed in the language of traditional analysis" was the second statement. The quaintness of those utterances in retrospect from 1979 came vividly to my mind when I was reading this book by R. Ansorge. Such a thorough and detailed investigation into the nature of finite difference methods would not now be considered a worthwhile effort if the first prediction had been right; and the book would not begin–as it does–with two sections entitled *Function-analytic formulation of initial value problems* and *The concept of a generalized solution*, if the language and methods of Functional Analysis had not, by now, deeply penetrated all work on partial differential equations.

There have been other widespread, more specific, predictions concerning trends in the numerical analysis of partial differential equations which would have pushed finite difference methods into the background, if they were true. One was that, as the available error estimates for these methods were, by necessity, always statements on the orders of magnitude only, rather than explicit realistic inequalities, they would be increasingly regarded as unreliable and worthless. Another one was the expectation that techniques of the Galerkin type, i.e., approximations in suitably constructed finite dimensional subspaces, such as those furnished by the finite element method would completely supersede the less flexible "old-fashioned" procedure of replacing derivatives by difference quotients in a grid.

For initial value problems, at least, as distinguished from boundary value problems, it is, however, still true that difference approximations are of paramount computational interest.

In the early history of this subject the name of Lewis F. Richardson stands out [5]. His grandiose scheme of an enormous staff of pencil pushing human computers numerous enough to solve with adequate accuracy the hyperbolic system of differential equations that governs the atmospheric flow of air was, of course, overly optimistic, as far as the mathematical difficulties were concerned and, at the same time, not imaginative enough to include the idea of machines each capable of replacing the arithmetic of millions of human beings.

As far as theory is concerned, Richardson was essentially satisfied with verification of the property now called *consistency*. This means that if $Fu = 0$ is the differential equation and F_h is a corresponding finite difference operator in a grid of meshlength h, then $\lim_{h\to 0}\{Fv - F_h v\} = 0$ for all functions v in a sufficiently smooth class. The study of the *convergence* of the solution of the difference equation problem to that of the underlying differential equation problem was taken up a few years later. The most important contribution is in the paper [1] by Courant, Friedrichs and Lewy. Among other seminal ideas it contains the discovery that consistency does not always imply convergence for problems of evolution type: Certain inequalities between the time mesh length $\Delta t = k$ and the space mesh $\Delta x = h$ have to be maintained in the passage to the limit, as $h \to 0$.

The third milestone in the history of the subject was J. v. Neumann's observation that certain procedures in which convergence, as explained above, is present are, nevertheless, practically useless, because they are not *computationally stable*. Convergence is a property of the exact solution of the finite difference problem. In practice, all computations have to be rounded off, however, and these "errors" often grow from one time step to another in an uncontrollable, frequently exponential manner. Such procedures are called computationally unstable [2], [6].

The relation between convergence and stability has been–and, as Ansorge's book shows, still is–an intriguing subject. A paper by Lax and Richtmyer [3] on this matter has been so important and influential that it deserves special mention as the fourth decisive step in the development of the subject. In particular it is the first contribution in which the reliance on functional analysis is more than decorative: Banach's "Theorem on uniform boundedness" is the main tool in the proof of a theorem to the effect that in the framework of properly chosen, quite natural definitions of these concepts convergence and stability *are* equivalent.

After the preceding historical sketch the contents of Ansorge's book can be described concisely: It explores the interrelation of the three concepts of consistency, convergence and stability in their dependence on the type of differential equation considered and on the particular way these three terms are defined.

The author is very systematic in the pursuit of his aim. Since the answers depend sensitively on the class of functions admitted and (less sensitively) on the metric in which the accuracy is measured, the above mentioned introductory sections on functiontheoretical formulations of the problems make possible a concise descriptive style. Without such a brief, precise terminology and a corresponding notation, the author's juxtaposition of no less than *nine* concepts of convergence would be hopelessly confusing. As it is, careful attention as well as good memory for definitions and symbols are demanded of the reader.

As every mathematician who has worked in this field well knows, one of its annoying, though superficial, difficulties is to find a notation that is not so complex as to obfuscate basically simple sequences of ideas. The author's struggle with this problem has been only partly successful. Some pages that are filled from top to bottom with lengthy formulas abounding in sub- and superscripts are rather discouraging. The unpleasantness is compounded by the fact that the book is in photooffset from a typescript rather than in ordinary print. On the other hand, the explanations are almost throughout careful and precise. One helpful feature is the presence of many supplementary explanatory remarks, which catch the eye, because they are printed in italics.

The author's main emphasis is the extension of methods based on the paper by Lax and Richtmyer to nonlinear problems. As the book progresses from linear problems to semilinear, quasilinear and completely nonlinear ones the statements of the theorems become, of necessity, more complicated, the hypotheses more restrictive and the proofs more involved. The formulations are, however, so general that they often also apply to integro-differential equations and problems that possess solutions only in a generalized sense. Surprisingly, the distinction between parabolic and hyperbolic problems appears hardly ever explicitly. Only a careful analysis of the hypotheses may reveal that they are sometimes satisfied for one type, but not for the other.

The large number of examples is very helpful. They are selected so as to illustrate a theoretical point as simply as possible rather than situations one is most likely to meet in computational applications.

In summary, this is a book for a theoretical numerical analyst, rather than for an applied scientist who wants to find quickly a good computational procedure for a practical problem at hand. The emphasis is on the structural relations between the continuous problem and the discrete ones that can be associated with it. The book collects the recent results in this direction from the widely scattered literature and presents them competently in a unified systematic way without making the mistake of aiming at a compendium of the subject.

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