Proposition 3.1 on p. 268, is a nonproof as far as I can see–if you know the formula, then it is true. It is too central to the theory of integration of forms to be treated this way.

The proof of the divergence theorem on p. 291 would be much simpler and clearer if the author used orthonormal moving frames, a topic I feel he does not exploit adequately in a number of places. Also, the messy calculation on p. 290, needed for the divergence theorem is only the obvious result that $f = \pm * d * \alpha$ follows from $f dV = d * \alpha$.

The discussion of Maxwell's equations on p. 329, as I read it, says that to prove Maxwell's equations it suffices to show that a certain 2-form ω is harmonic (see lines 6*–5*). This means, in harmonic form language, that $\Delta\omega = 0$ implies $d\omega = 0$ and $\delta\omega = 0$, a true statement in *compact* manifolds (p. 320), but not applicable here. What is more, the issue is clouded because the Hodge theory (and indeed almost everything else metric in the book) is stated for Riemannian manifolds, whereas here we are dealing with a Lorentz metric (invariant $+, -, -, -$). This is an alert for those attempting the last chapter of the book, *General theory of relativity*. The author sometimes assumes that facts about Riemannian manifolds carry over automatically to the Lorentz (pseudo-Riemannian) case. Sometimes they do, sometimes they don't.

I conclude that someone who is reasonably familiar with the mathematics of this book will be able to get something out of the applications to physics, provided he works at it harder than he should have to and doesn't accept the author's mathematics at face value. I doubt very much if someone can learn the subject from this book without extensive work in other sources. (Also the author gives no exercises.) The book may repel rather than attract, exactly the opposite of its author's intent. Too much of the book was written without adequate organization and care, and I get the feeling that some sections must have been written at wide time intervals. It's a pity that the author didn't make a more palatable product. The subject will eventually be a standard tool in physics, but there is yet little material accessible to nonmathematicians. My own short book [1] doesn't begin to compare in its scope of applications to the book under review because I didn't when I wrote it and never will know physics as Profesor von Westenholz does.

References

1. H. Flanders, *Differential forms with applications to the physical sciences*, Academic Press, New York, 1963.

2. C. Godbillon, *Géométrie différentielle et méchanique analytique*, Hermann, Paris, 1969.

3. R. Hermann, *Differential geometry and the calculus of variations*, Academic Press, New York, 1968.

HARLEY FLANDERS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 6, November 1979

0002-9904/79/0000-0512/$01.75

Combinatorial group theory, by Roger C. Lyndon and Paul E. Schupp, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 89, Springer-Verlag, Berlin and New York, 1977, ix + 339 pp.

Combinatorial group theory: *presentations of groups in terms of generators and relations* by Magnus, Karrass and Solitar was published in 1966. It was a careful, leisurely exposition of some of the basic topics in a loosely defined area, with close connections to topology and logic, aimed specifically at beginning graduate students which has now found a much wider use. The book under review by Lyndon and Schupp is similarly entitled *Combinatorial group theory*. In a sense the similarity ends with the title. For in keeping with the spirit of the Ergebnisse series, this book is for the most part a fast moving, technical exposition, packed with information about a field which has mushroomed in the past 30 years.

The origins of combinatorial group theory can be traced back to the middle of the 19th century. With surprising frequency problems in a wide variety of disciplines, including the theories of differential equations and automorphic functions were distilled into questions about explicit groups. These groups took on many forms–matrix groups, discontinuous groups of transformations, symmetry groups, groups preserving, e.g. quadratic forms and numerous others. The introduction of the fundamental group by Poincaré in 1895, the discovery of knot groups by Wirttinger in 1905 and the proof by Tietze in 1908 that the fundamental group of a compact finite dimensional arcwise connected manifold is finitely presented served to underline the importance of finitely presented groups. This was followed by a sequence of papers between 1910 and 1914 by Max Dehn who explicitly raised (and partly solved) a number of fundamental problems about finitely presented groups, thereby heralding the birth of a new subject–combinatorial group theory. Thus the subject came endowed and encumbered by many of the problems that had stimulated its birth.

These problems were generally concerned with various classes of groups and were of the following kind: Are all of the groups in a given class finitely generated? Finitely presented? Finite? What are the conjugates of a given element in a given group? What are the subgroups of that group? Is there an algorithm for deciding for every pair of groups in a given class whether or not they are isomorphic? And so on. The objective of combinatorial group theory is the systematic development of algebraic techniques to settle such questions. In view of the scope of the subject and the extraordinary variety of groups involved it is not surprising that no really general theory exists. It is however surprising that so much has been accomplished and that so many methods and techniques have been developed with such wide application and potential. Some of these techniques have even found a wider use in e.g., ring theory, in logic and even in topology itself. It seems appropriate to discuss here in greater detail only those techniques and ideas dealt with by Lyndon and Schupp.

In 1921 Nielsen developed a method for studying the subgroups of free groups. His procedure was to transform, in a systematic manner, a given set of generators of a subgroup of a free group (equipped with a fixed free set of generators) into another set of generators which has the property that very little cancellation takes place between these new generators. This method yields in particular, a proof that the subgroups of a (finitely generated) free group are again free. Its scope, however, is by no means restricted to free groups. Indeed the combination of related ideas and primitive combinatorial geometric methods has given rise to so-called small cancellation theory, anticipated by Dehn himself. The book by Lyndon and Schupp contains a careful, detailed account of this theory, which is today a powerful tool in combinatorial group theory, of interest in its own right. It has led to the solution of problems which were seemingly intractable only a few years before. A stunning illustration is the construction by Shelah of an uncountable group all of whose proper subgroups are countable.

Nielsen used his method to prove, in 1924, that the automorphism group of a free group of finite rank is finitely presented. This was followed in 1936 by deep work of J. H. C. Whitehead who proved that it is possible to decide whether there is an automorphism α of a free group F which takes one finite sequence $(a_1, \ldots, a_n)$ of elements of F onto another such sequence $(b_1, \ldots, b_n)$ (i.e., such that $a_i\alpha = b_i$ $(i = 1, \ldots, n)$). Whitehead's proof was topological. It was not until 1958 that an algebraic proof was obtained by Rapaport (= Strasser) which was simplified and improved by Higgins and Lyndon in 1974. The importance of Whitehead's ideas has most recently been underlined by some very fine work of McCool who proved in particular, in 1975, that the group of automorphisms of a free group F of finite rank which map a finite subset of F onto itself, is finitely presented. This is reminiscent of a well-known theorem of Borel and Harish Chandra which asserts that rational arithmetic groups are finitely presented.

Free products with one amalgamated subgroup were invented by Schreier in 1927. They have since turned out to be one of the most important constructive tools in combinatorial group theory with applications to logic and to topology as well as to group theory itself. Indeed Magnus recognized their scope in 1932 in his pioneering work on groups with a single defining relation. The subgroups of free products, without amalgamations, were determined by Kurosh in 1934. This was followed in 1948/49 by a vastly more general but less explicit theorem by Hanna Neumann on the subgroups of so-called generalized free products. In 1949 G. Higman, B. H. and Hanna Neumann used generalized free products to prove certain embedding theorems and in particular that every countable group is a subgroup of a two-generator group. Their proof introduced a new construction, based on the generalized free product of two groups with one amalgamated subgroup, now termed an *HNN*-extension, which has found wide application. In view of the importance of generalized free products the explicit determination, in 1970, by Karrass and Solitar of the subgroups of a free product of two groups with one amalgamated subgroup represented a major step forward. This was preceeded by earlier work of Serre in 1968/69, who introduced a theory of groups acting on trees. This work of Serre was amplified and extended by Bass. The net result has been the creation of a general theory of quite general use. It should be noted that the spirit of Serre's approach is topological, motivated by the theory of covering spaces.

Covering space theory has long been an important contributor to combinatorial group theory. Various attempts to express the ideas in this theory in purely graph-theoretic terms have been made from time to time with varying degrees of success. The aim is to replace analytic and topological arguments by combinatorial geometric ones. This was carried out in detail by Reidemeister in 1932 who obtained graph-theoretic proofs of the classical classification of compact two-dimensional surfaces. These methods have a certain elegance yielding clean proofs of, in the main, well-known theorems. This approach is taken up in some detail by Lyndon and Schupp leading naturally to relationships between presentation theory and cohomology, to various geometric representations of groups as well as to a technique of Behr for proving that certain groups of isometrics are finitely presented.

Covering space techniques are particularly useful in the proofs of subgroup theorems. There is, however, a simple algebraic way of proving that the subgroups of a free group are again free, due to Schreier. Schreier's proof has other advantages in that it provides a method for finding generators and defining relations for a subgroup of a group given by generators and defining relations (the so-called Reidemeister-Schreier method). It was, in part, on this technique that Magnus relied to solve the word problem for one-relator groups. It has also been extremely useful in many other situations. However it is not sufficiently incisive to be able to give really precise information about the subgroups of finitely presented groups. In fact, despite the continuing demands from other disciplines for more detailed knowledge of the subgroup structure, no general results were known before 1960. Then in 1961, in one extraordinary stroke, G. Higman explained the reason for this lack of progress, uncovering a link between finitely presented groups and recursive functions. Indeed Higman completely determined the finitely generated subgroups of finitely presented groups by identifying them as the finitely generated groups that can be defined by a recursively enumerable set of defining relations. Thus the theory of the finitely generated subgroups of finitely presented groups is intimately related to that of recursive functions. Higman's theorem then yields easily the existence of a host of finitely presented groups with unsolvable word problems. The first example of such a group was constructed, with enormous difficulty, by Novikov in 1955. Indeed Novikov pointed out also at that time that the existence of such a group implies that the isomorphism problem for finitely presented groups is algorithmically unsolvable. Indeed Adyan in 1956 was the first to prove that "almost all" problems about finitely presented groups are algorithmically unsolvable, including for example the triviality problem: there is no general and effective procedure whereby one can determine whether any finitely presented group is of order one! The proof of Higman's theorem as detailed by Lyndon and Schupp is due to Valiev, depending on the characterization by Matiyasevich of recursively enumerable subsets of the nonnegative integers as the zeros of appropriately chosen diophantine equations.

Much of this and more is contained in the book by Lyndon and Schupp, a book well-worth reading, a book well-worth buying. The proof of Whitehead's theorems, and the related theorems of McCool, the proof of the Karrass-Solitar theorem on the subgroups of free products with one amalgamated subgroups by Nielsen techniques and its obvious promise of applications, the discussion of cohomology, the graph-theoretical connections, the discussion of *HNN* extensions, the elegant treatment of one-relator groups, the proof of the Higman embedding theorem, the connections with logic, the use of van Kampen diagrams and the treatment of small cancellation theory and its applications represent very fine achievements. Much of the material has appeared only in the periodical literature until now; indeed some of the material appears here in print for the first time. The book is clearly an important contribution to the mathematical literature. But it is only fair that I add some words of warning. The authors have followed their personal interests a little too closely. As a consequence the broad scope of the subject itself is only hinted at. The book was written in two parts, the first by one author, the second by the other, and common material was simply repeated as it arose. Apparently this was intentional, allowing the reader to read each chapter as a separate entity. Nevertheless the arrangement of the material is haphazard, the exposition is very uneven, some of it is unnecessarily hard to follow, some almost impossible. There are much too many misprints, successive paragraphs are sometimes unrelated and motivation is almost totally lacking. Some of the text has not been well-worked out, the graph-theoretic-topological parts demand varying levels of topological expertise and no attempt has been made to find the general topological principles that govern much of this material as well as many of the subgroup theorems. The notion of an aspherical presentation is somehow identified with the topological notion of asphericity without sufficient justification. In spite of these very real criticisms this is still an important piece of work.

GILBERT BAUMSLAG

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 6, November 1979

0002-9904/79/0000-0513/$02.75

Bounded integral operators on L^2 spaces, by P. R. Halmos and V. S. Sunder, Ergebnisse der Mathematik und ihrer Grenzgebiete no. 96, Springer-Verlag, Berlin and New York, 1978, xv + 134 pp., $16.50.

This slim volume in the Ergebnisse series (to which I shall refer as H-S) deals with bounded integral operators on L^2 spaces, that is to say, bounded linear operators $K: L^2(Y, \nu) \to L^2(X, \mu)$ of the form

$$(Kf)(x) = \int_Y k(x, y) f(y)\, d\nu(y)$$

for all $f \in L^2(Y, \nu)$, where (X, μ) and (Y, ν) are σ-finite and separable measure spaces and the integral is an "ordinary" one with respect to ν (no principal values; no L^2 limits as in the theory of Fourier transforms). Restriction to a σ-finite separable measure implies (by a well-known isomorphism theorem) that for most purposes it may just as well be assumed that ν is either Lebesgue measure in the interval [0, 1] or counting measure in **Z** (or **N**) or a finite subset of **N**. There are two recent books on integral operators of a general nature (i.e., not restricted to L^2 operators), one by the late K. Jörgens ([**2**, 1970], in German) and one by M. A. Krasnosel'skiĭ, P. P. Zabreĭko, E. I. Pustyl'nik and P. E. Sobolevskiĭ ([**3**, 1966], in Russian; English translation 1976). We may ask, therefore, if it is still possible to say something of interest about the simple L^2 case that has not been said many times before.