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STABLE COMPLETE MINIMAL SURFACES IN R^3 ARE PLANES

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ABSTRACT. A proof of the statement in the title is given.

1. Introduction.

1.1 Let M be a two-dimensional, orientable, connected manifold and let $x: M \longrightarrow R^3$ be a minimal immersion of M into the euclidean space R^3. A domain $D \subset M$ with compact closure is stable if the second variation of the induced area of D is nonnegative for all variations that leave the boundary ∂D of D fixed. The immersion x is *stable* if every such D is stable. The aim of this note is to prove the following result.

1.2 THEOREM. *Let $x : M \longrightarrow R^3$ be a stable complete minimal immersion. Then $x(M) \subset R^3$ is a plane.*

Theorem 1.2 is a generalization of the Bernstein theorem which states that if $x(M)$ is a complete minimal graph (minimal graphs are stable), then $x(M)$ is a plane. With the additional condition that the total curvature is finite, the theorem has been proved by M. do Carmo and A. M. da Silveira in [2] and later extended to the case when the total curvature has a small order of growth by M. do Carmo and C. K. Peng in [3]. We have been informed by R. Schoen that, in a joint work with D. Fischer-Colbrie, he also obtained a proof of Theorem 1.2 (see [4]). We wish to thank S. S. Chern, B. Lawson and S. T. Yau for their interest in this work.

2. Proof of Theorem 1.2.

2.1 We first show that we can restrict ourselves to the universal covering $\pi: \widetilde{M} \longrightarrow M$ of M. Explicitly, we show that if there exists an unstable relatively compact domain $\widetilde{D} \subset \widetilde{M}$ for the immersion $x \circ \pi: \widetilde{M} \longrightarrow R^3$ then $\pi(\widetilde{D}) \subset M$ is unstable. We will denote by Δ_M, ∇_M and K, respectively, the Laplacian, the gradient and the Gaussian curvature in the induced metric of M.

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If $\tilde{D}$ is unstable, there exists a domain $\tilde{D}' \subset \tilde{D}$ and a nonnegative function u in $\tilde{D}'$ that vanishes on $\partial\tilde{D}'$ and satisfies $\Delta_M u - 2Ku = 0$ in $\tilde{D}'$. Extend u to be zero outside $\tilde{D}'$. For each $q \in M$, let $\pi^{-1}(q) = \{q_1, \ldots, q_k, \ldots\}$ and define a function f on M by $f(q) = \Sigma_i u(q_i)$. Since the closure of $\tilde{D}'$ is compact this sum is nonzero only for finitely many indices. Set $\pi(\tilde{D}') = D'$. It is not hard to show that f is continuous on M, $f \geq 0$, $f \equiv 0$ on $\partial D'$ and $\int_{\bar{D}'} |\nabla f|^2\, dM \leq 2\int_{\bar{D}'}(-K)f^2\, dM$ (the proof is essentially contained in [**1**, pp. 521–526]). It follows that D contains a conjugate boundary, hence is unstable as we wished to show.

Thus we can assume that M is simply-connected. With the natural complex structure given by x, M is then conformally equivalent to either the complex plane **C** or the unit disk $B = \{z \in \mathbf{C};\ |z| \leq 1\}$, and the induced metric ds^2 in M is given by $ds^2 = \lambda^2|dz|^2$, $\lambda \neq 0$.

2.2 Let us first consider the case of the unit disk B. By assuming that every relatively compact subdomain $D \subset M$ is stable, we have that

$$\int_M (u\Delta_M u - 2u^2 K)\, dM \leq 0, \tag{2.3}$$

for all piecewise smooth functions u that are compactly supported in M. Let Δ denote the Laplacian and dA the element of area in the flat metric. Then

$$K = -\frac{1}{\lambda^2}\Delta\log\lambda, \quad dM = \lambda^2\, dA, \quad \Delta_M = \frac{1}{\lambda^2}\Delta,$$

and (2.3) can be written as

$$\int_B (u\Delta u + u^2\Delta\log\lambda^2)\, dA \leq 0. \tag{2.4}$$

By setting $\varphi = \lambda^{-1}$ and replacing in (2.4) u by φu, we obtain

$$3\int_B |\nabla\varphi|^2 u^2\, dA \leq \int_B \varphi^2|\nabla u|^2\, dA - 2\int_B \varphi u(\nabla u \cdot \nabla\varphi)\, dA. \tag{2.5}$$

Since, for any $\epsilon > 0$,

$$|\varphi u \nabla u \cdot \nabla\varphi| \leq \epsilon|\nabla\varphi|^2 u^2 + \frac{1}{\epsilon}\varphi^2|\nabla u|^2,$$

(2.5) implies that there exists a constant $\beta > 0$ such that

$$\int_B |\nabla\varphi|^2 u^2\, dA \leq \beta\int_B \varphi^2|\nabla u|^2\, dA,$$

which finally gives, since $\nabla_M = (1/\lambda)\nabla$,

$$\int_M |\nabla_M\varphi|^2 u^2\, dM \leq \beta\int_M \varphi^2|\nabla_M u|^2\, dM. \tag{2.6}$$

Now choose a family of geodesic balls B_R of radii R that exhausts M, fix θ, $0 < \theta < 1$, and let $u\colon M \to R$ be the continuous function that is one on $B_{\theta R}$, zero outside B_R and linear in $B_R - B_{\theta R}$. From (2.6) we obtain that

$$\int_{B_R} |\nabla_M \varphi|^2 \, dM \leqslant \frac{\beta}{(1-\theta)^2 R^2} \int_M \varphi^2 \, dM$$

$$= \frac{\beta}{(1-\theta)^2 R^2} \int_B dA = \frac{\pi\beta}{(1-\theta)^2 R^2}.$$

By letting $R \longrightarrow \infty$, we conclude that $|\nabla \varphi| \equiv 0$, i.e., $\lambda = \text{const}$, and this contradicts the completeness of $ds^2 = \lambda^2 |dz|^2$.

2.7 Let us now consider the case where M is conformally equivalent to the complex plane $\mathbf{C}$. There are several ways of working out that case, the shortest one being probably as follows. By setting $\psi = \Delta \log \lambda^2$, we can write (2.4) as

$$\int_{\mathbf{C}} \psi u^2 \, dA \leqslant \int_{\mathbf{C}} |\nabla u|^2 \, dA. \tag{2.8}$$

On the other hand, if K is not identically zero, we know that (cf. [3, Remark 2]). $\Delta_M \log(-K) = 4K$. This implies that $\Delta \log \psi + \psi = 0$, hence

$$\psi \Delta \psi + \psi^3 = |\nabla \psi|^2. \tag{2.9}$$

From that point on, the proof is the same as in [3, §2] and we give only an outline of it.

By replacing u by ψu in (2.8), we obtain

$$\begin{aligned} \int_{\mathbf{C}} \psi^3 u^2 \, dA &\leqslant \int_{\mathbf{C}} \psi^2 |\nabla u|^2 \, dA \\ &+ \int_{\mathbf{C}} u^2 |\nabla \psi|^2 \, dA + 2 \int_{\mathbf{C}} \psi u \nabla u \cdot \nabla \psi \, dA. \end{aligned} \tag{2.10}$$

On the other hand, by multiplying (2.9) by u^2, integrating over $\mathbf{C}$ and adding up the result to (2.10), one obtains

$$\int_{\mathbf{C}} |\nabla \psi|^2 u^2 \, dA \leqslant \int_{\mathbf{C}} \psi^2 |\nabla u|^2 \, dA. \tag{2.11}$$

By using in the last summand of (2.10) the fact that $2ab \leqslant \epsilon a^2 + (1/\epsilon) b^2$, for all $\epsilon > 0$, and introducing (2.11) into (2.10), we obtain

$$\int_{\mathbf{C}} \psi^3 u^2 \, dA \leqslant \beta_1 \int_{\mathbf{C}} \psi^2 |\nabla u|^2 \, dA, \quad \beta_1 = \text{const}. \tag{2.12}$$

Now use in (2.12) Young's inequality,

$$\psi^2 |\nabla u|^2 = u^2 \left(\psi^2 \frac{|\nabla u|^2}{u^2} \right) \leqslant u^2 \left(\frac{\alpha^s}{s} \psi^{2s} + \frac{\alpha^{-t}}{t} \left(\frac{|\nabla u|}{u} \right)^{2t} \right),$$

which holds for all $\alpha > 0$ and all $1 < s, t < \infty$, with $(1/s) + (1/t) = 1$. Choose

$s = 3/2$, $t = 3$ and α small to obtain a constant β_2 so that

$$\int_{\mathbf{C}} \psi^3 u^2 \, dA \leqslant \beta_2 \int_{\mathbf{C}} \frac{|\nabla u|^6}{u^4} \, dA,$$

By changing u into u^3 into the above inequality, we finally arrive at

$$\int_{\mathbf{C}} \psi^3 u^6 \, dA \leqslant \beta_3 \int_{\mathbf{C}} |\nabla u|^6 \, dA, \qquad \beta_3 = \text{const.} \tag{2.13}$$

Inequality (2.13) implies, by choosing the usual u in a ball $B_R \subset \mathbf{C}$ of radius R, that $\psi^3 \equiv 0$, hence $K \equiv 0$ and $x(M)$ is a plane. This concludes the proof of Theorem 1.2.

REFERENCES

1. J. L. Barbosa and M. do Carmo, *On the size of a stable minimal surface in R^3*, Amer. J. Math. 98 (1976), 515–528.

2. M. do Carmo and A. M. da Silveira, *Globally stable complete minimal surfaces in R^3*, Proc. Amer. Math. Soc. (to appear).

3. M. do Carmo and C. K. Peng, *Stable complete minimal hypersurfaces* (to appear).

4. D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature* (preprint).

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