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Infinite loop spaces, by J. F. Adams, Ann. of Math. Studies no. 90, Princeton Univ. Press, Princeton, N. J., 1978, x + 214 pp., $14.00 (cloth), $5.50 (paper).

Algebraic topology is a very young and restless subject. Many of its most active areas did not exist a decade ago, or existed only in embryonic form. Examples include the theory of localizations and completions, rational homotopy theory, the study of finite H-spaces, exploitation of the Brown-Peterson spectrum and other new techniques in the calculation of stable homotopy groups, algebraic K-theory and the homotopy theory of categories, exploitation of techniques from algebraic geometry, and infinite loop space theory and its applications. Similarly, many techniques and constructions that are already accepted by workers in the field as standard and elementary are equally new. One thinks of localizations, completions, the classifying spaces of monoids and categories, the geometric transfer, the plus construction, etc. Not one of these things is so much as mentioned in even the most recent and advanced texts in the subject (and if this sounds like a rebuke to their authors, why so be it).

If this is true of standard and elementary parts of the subject, then it is hardly surprising that the deeper machinery relevant to the more sophisticated new areas is virtually inaccessible without direct contact with practitioners. The goal of Adams' book is to "convey the basic ideas of the subject in a way as nearly painless as I can make it". By "the subject" he means infinite loop space theory. But in fact he has succeeded in giving the basic ideas not just of this specialty but of much of modern algebraic topology, including capsule introductions to many of the topics mentioned above. I urge anybody teaching algebraic topology on any level and anybody working or thinking of working in the subject to read this book. One or two patches might be a little hard going, particularly in Chapter 6 (and the reader is given fair warning), but for the most part the book provides some of the most delightful and illuminating exposition to be found, not just in topology, but in mathematics. It is written with style and wit, and reads like a novel (in places, as on pp. 112 and 144, like a roman a clef, although the characters in the drama are usually identified even when being chided). The truth is, whatever we may say, that we invent mathematics because it is fun. Seldom has the spirit of the enterprise been so successfully captured in print. Would that treatments in a similar vein were available in other supposedly impenetrable abstract areas of mathematics.

Nevertheless, of course, the emphasis is on infinite loop space theory, and some of my colleagues don't much see the fun in that. There are various areas of mathematics that are widely regarded with suspicion and dislike because of the quantity of pure abstraction involved. There are also various areas that are widely regarded with suspicion and dislike because of the quantity of grubby calculation involved. Infinite loop space theory runs simultaneously to both extremes and so has something to offend everyone.

What cannot be denied is that the calculations work. For example, if one wants to know the characteristic classes for topological bundles or for spherical fibrations or, more concretely, if one wants to determine whether a given Poincaré duality space has the homotopy type of a topological manifold or if one wants to determine the set of equivalence classes under cobordism of topological manifolds, then one must learn something about infinite loop space theory.

These are only a few of the earlier calculational applications. There are various other direct and indirect applications to manifold theory, algebraic K-theory, and stable and unstable homotopy theory, and the number of applications is growing rapidly.

Adams' book concentrates on the theoretical core of the subject. He cites **[14]** for a more complete survey of the results and applications. However, although written in 1976, that reference is already woefully incomplete. The three articles **[15]**, **[16]**, and **[17]** summarize more recent developments and together include sixty or seventy references too new to appear in the bibliographies of **[14]** or of Adams' book. I must apologize for referring to so much of my own writing but, despite the very large amount of work going on in this area, there are no other surveys except the brief early one of Stasheff **[23]**. Comprehensive treatments of the earlier applications and calculations are to be found in **[4]**, **[13]**, and Madsen and Milgram **[9]**.

Well, what is infinite loop space theory? There are three ways of looking at what amounts to more or less the same subject, displayed as

$$\text{cohomology theories} \leftrightarrow \text{spectra} \leftrightarrow \text{infinite loop spaces.} \qquad (*)$$

A cohomology theory E^* on spaces assigns a group E^nX to each space X in such a way that the Eilenberg-Steenrod axioms other than the dimension axiom are satisfied. Then E^nX is a "representable functor" of X, so that E^nX is the group of homotopy classes of maps $X \to E_n$ for some space E_n. The axioms imply that E_n is equivalent to ΩE_{n+1}, the "loop" space of (based) maps from a circle into E_{n+1}. Spaces such as E_0 are called infinite loop spaces, meaning that there is a space E_n and an equivalence of E_0 with $\Omega^n E_n$ for each $n \geqslant 0$.

The core of infinite loop space theory is concerned with the elucidation of the internal algebraic structure present on infinite (and n-fold) loop spaces. These are H-spaces with all sorts of infinite families of coherence homotopies for associativity and commutativity of their products, and a major task is the encapsulation of all this information in workable form. With a suitable formulation, one can then reverse the passage from spectra $\{E_n\}$ to infinite loop spaces E_0. That is, given a suitably structured space X, one can manufacture "deloopings" B^nX and equivalences between X and $\Omega^n B^n X$. There are three main machines for carrying out this program, due to Boardman and Vogt, Segal, and myself **[3]**, **[22]**, **[12]**. All three may be viewed as exercises in topological algebra, the study of algebraic structure on topological spaces. However, the elaborate nature of the algebraic structure that must be encoded demands an abstract context of categorical topological algebra previously foreign to algebraic topology. Adams gives motivation for and incisive descriptions of these machines. At the very end, he mentions the recent work of Thomason and myself [18] which axiomatizes the passage from structured spaces to spectra and so shows that there is really only one machine, but this development came too late to affect the body of the book.

One virtue of studying such structured spaces is that the internal algebraic structure leads directly to powerful computable invariants, notably homology operations and transfer, the latter of which is quite thorougly studied in Adams' book. (Cohen and I have given a comprehensive treatment of the former [4].) For the applications, the crucial point is that, while the three concepts in the display (*) are essentially equivalent from a categorical point of view, they carry quite different and mutually complementary calculational information.

As I have already mentioned, exploitation of this structure now impinges on a broad range of topological problems far removed from the concerns of infinite loop space theory. I would like to reinforce this point by citing a few quite recent examples.

This structure leads (in [12]) to simple combinatorial approximations to spaces of the form $\Omega^n\Sigma^n X$. Mahowald [11] found a remarkable way to exploit this structure to construct infinite families of elements in the stable homotopy groups of spheres. This has led to a stream of further work, and there are other quite different ways (summarized in [17]) of exploiting the cited approximation. In particular, it leads to a very simple proof of the Kahn-Priddy theorem [7] to the effect that the p-torsion of the stable homotopy groups of spheres is a direct summand of the stable homotopy groups of the classifying space of the pth symmetric group. Infinite loop space techniques in one form or another are essential to any proof of this result. In turn, this result and further techniques from infinite loop space theory are essential to Nishida's proof [20] (see also [15]) of the nilpotency of the ring of stable homotopy groups of spheres.

Again, infinite loop techniques play a key role in Waldhausen's marvelous theory [25] (see also Steinberger [24]) relating the stable concordance groups of PL manifolds to algebraic K-theory.

In connection with algebraic K-theory, Quillen's work [21] relating the general linear groups of finite fields to the "Image of J" spaces of topological K-theory extends to all classical groups of finite fields and leads to a rich feedback circuit relating these finite groups and their homologies to the infinite loop spaces of geometric topology and their homologies [Fiedorowicz and Priddy [5], [4], [13], and [14]].

In these theories, a ring theoretical elaboration [13] of the topological algebra of infinite loop space theory plays a central role.

Another example, discussed in Adams' book, is Becker and Gottlieb's striking use [2] of the infinite loop space structure on the classifying space for stable spherical fibrations in their proof of the Adams conjecture.

As a final example, I cite Jones' use [6] of Kochman's calculations [8] of the homology operations of classical groups in his concrete construction of a 30 dimensional manifold with nontrivial Kervaire invariant.

What these applications have in common–and the list is far from exhaustive–is the use of the results but not of the internal machinery of infinite loop space theory. It is a sign of a genuinely important technique in algebraic topology that it becomes a standard every day tool used by workers quite unconcerned with how the tool was derived. It is a peculiarity of the subject that the work involved in setting up machinery–whether it be singular homology, the Serre or any other spectral sequence, localization and completion, or the machinery under discussion–often bears little or no technical relationship to the work involved in applying the machinery. Adams' book, while also succeeding in the wider (and perhaps unintended) goal of introducing much of modern algebraic topology, is primarily aimed at allowing present and prospective workers to feel at home with the new infinite loop space machinery. It succeeds admirably.

A secondary and more technical goal, limited to Chapter 6, is the detailed analysis of the relationship between maps of the spectra of topological K-theory and maps of their underlying infinite loop spaces. This builds on work of Adams and Priddy **[1]**, which is summarized, and is primarily due to Madsen, Snaith, and Tornehave **[10]**, whose main theorems are given complete new proofs.

Of course, a few quibbles are de rigeur. Adams rightly emphasizes the "group completion theorem " describing the behavior of the natural map $G \to \Omega BG$ for a not necessarily grouplike topological monoid G. This result is crucial to all rigorous verifications of the central axiom given in **[18]** for infinite loop space machines. However, in Adams' sketch proof, it is assumed that "the technicians" can define a certain comparison map of spectral sequences. Perhaps they can (although this one needed help from Adams), but the audience for whom the book is intended surely cannot. To my mind, the definitive treatment of this result is in the paper **[19]** of McDuff and Segal.

Again, the particular geometric construction of the transfer given in §4.1 strikes me as redundant. The construction in §4.2 seems much more useful and is the one relevant to later parts of the book. While Adams does use the first description to derive properties of the transfer in §4.3, the simpler verifications work equally well either way, while the product formula left unfinished on page 127 comes much more simply from the second description.

And some of the jokes (listed as such in the index!) are excruciating.

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Hilbert's third problem, by Vladimir G. Boltianskiĭ (translated by Richard A. Silverman and introduced by Albert B. J. Novikoff), Scripta Series in Math., Wiley, New York, 1978, x + 228 pp., $19.95.

1. Since the response to the title of this book is invariably "What *is* Hilbert's third problem?", let us begin by considering the problem itself. Loosely speaking, it asks whether there is any way of deriving the formula for the volume of a tetrahedron without using calculus. Clearly there is no hope of avoiding all mention of limits in most questions of volume, for it is by appealing to a limit process that the very notion of volume is extended to any figure more general than a rectangular solid having rational edges. Analogously, limits are needed to extend the concept of area beyond rectangles having rational sides. Hilbert's problem acknowledges such fundamental