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FINITENESS THEOREMS FOR POLYCYCLIC GROUPS

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Introduction. A group G is *polycyclic* if it is built up from the identity by finitely many successive extensions with cyclic groups, or equivalently if it is isomorphic to a soluble group of matrices over $\mathbf{Z}$ (not obvious!). The second definition makes it clear that the normal subgroups of finite index in G intersect in 1, so one may hope that the finite quotient groups of G will carry a lot of information about the structure of G. The first main result says that in fact they "almost" determine G up to isomorphism, i.e. they do so up to finitely many possibilities. (Examples show that there really are finitely many possibilities, not just one.) The second main result is a sort of "concrete" analogue of this: if G is contained in $GL_n(\mathbf{Z})$, then there are only finitely many possibilities up to conjugacy in $GL_n(\mathbf{Z})$ for subgroups H in $GL_n(\mathbf{Z})$ such that H is "conjugate to G modulo m" for all nonzero integers m. This is related to classical results in arithmetic, like the fact that there are only finitely many inequivalent integral quadratic forms with given determinant, and the Hasse-Minkowski Theorem.

Results. Denote by $\mathfrak{F}(G)$ the set of isomorphism classes of finite quotients of a group G, and by $\hat{G}$ the profinite completion of G. For polycyclic-by-finite groups G and H, $\mathfrak{F}(G) = \mathfrak{F}(H)$ if and only if $\hat{G} \cong \hat{H}$; if this holds we say that G and H belong to the same $\hat{}$-class.

THEOREM 1. *Every $\hat{}$-class of polycyclic-by-finite groups is the union of finitely many isomorphism classes.*

A major ingredient in the proof of this is a result about arithmetic groups. Let G be an algebraic matrix group defined over $\mathbf{Q}$, and denote by π_m: $G(\mathbf{Z}) \longrightarrow G(\mathbf{Z}/m\mathbf{Z})$ the canonical map. For subgroups X and Y of $G(\mathbf{Z})$, say $X \sim_G Y$ if for every $m \neq 0$, $X\pi_m$ and $Y\pi_m$ are conjugate in $G(\mathbf{Z}/m\mathbf{Z})$.

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THEOREM 2 (F. J. G. and D. S.). *Every $\sim_G$-class of soluble-by-finite subgroups of $G(\mathbf{Z})$ is the union of finitely many conjugacy classes in $G(\mathbf{Z})$.*

Special cases of Theorem 1 have appeared in [P1], [P2], [GS1], and a special case of Theorem 2 in [GS1].

AUXILIARY RESULTS. We state now some further results used in the proofs.

THEOREM 3 [S]. *If G is a polycyclic group and d is a positive integer, then up to isomorphism there are only finitely many extensions of G by a group of order d.*

This is needed for Theorem 1. The next three results are needed for Theorem 2.

THEOREM 4 [S]. *If G is an algebraic matrix group defined over $\mathbf{Q}$ and X is a soluble subgroup of $G(\mathbf{Z})$, then the subgroups of $G(\mathbf{Z})$ which contain X as a normal subgroup of finite index lie in finitely many conjugacy classes in $G(\mathbf{Z})$.*

Let $G \leqslant GL_n$ be an algebraic matrix group defined over $\mathbf{Q}$, and let π_m now denote the canonical map $\mathbf{Z}^n \longrightarrow (\mathbf{Z}/m\mathbf{Z})^n$. For $\mathbf{Z}$-submodules A and B of $\mathbf{Z}^n$, say $A \sim_G B$ if for every $m \neq 0$, $A\pi_m$ and $B\pi_m$ lie in the same orbit of $G(\mathbf{Z}/m\mathbf{Z})$.

THEOREM 5 [GS2]. *Every $\sim_G$-class of $\mathbf{Z}$-submodules of $\mathbf{Z}^n$ is the union of finitely many orbits of $G(\mathbf{Z})$.*

For the next result, let $\mathfrak{o}$ be the ring of integers in an algebraic number field and denote by $\mathcal{P}$ the set of all nonzero prime ideals of $\mathfrak{o}$. Call a subset $\mathcal{Q}$ of $\mathcal{P}$ *ample* if every subgroup of finite index in the units group $\mathfrak{o}^*$ of $\mathfrak{o}$ contains a subgroup of the form $(1 + \mathfrak{a}) \cap \mathfrak{o}^*$ where $\mathfrak{a}$ is an intersection of members of $\mathcal{Q}$.

THEOREM 6 [GS3]. *If $\mathcal{F}$ is a finite subset of $\mathcal{P}$ and $\mathcal{P} - \mathcal{F}$ is partitioned into finitely many subsets, then at least one of these subsets is ample.*

OUTLINE PROOF OF THEOREM 1. Consider a set $\mathcal{C}$ of polycyclic-by-finite groups, contained in a single $\hat{}$-class. By Theorem 3 we may assume that for each $G \in \mathcal{C}$, the Fitting subgroup N_G of G is torsion-free and G/N_G is free abelian. Since [P3] $\widehat{N_G}$ is the Fitting subgroup of $\hat{G}$, we may apply the special case of Theorem 1 for nilpotent groups [P1] and assume that the groups N_G for $G \in \mathcal{C}$ are all isomorphic. We then use Theorem 2, applied to the arithmetic group $\mathrm{Aut}(N_G)$, to reduce to the case where the action of G on N_G is the same for all $G \in \mathcal{C}$; i.e. the pairs $(G/\zeta_1(N_G), N_G)$ are all isomorphic. Write Q_G for the hypercentre of G. Using a cohomological result due to Robinson [R] one can further reduce to the situation where the groups G/Q_G are all isomorphic, compatibly with the isomorphisms linking the various N_G.

The idea is now to form *semisimple splittings* of the groups in $\mathcal{C}$ (see [T]). For each G we construct an abelian subgroup $T_G \leqslant \mathrm{Aut}(G)$ such that the split extension $G\,]\,T_G$ is equal to $M_G\,]\,T_G$ where M_G is the Fitting subgroup of $G\,]\,T_G$. Then $G\,]\,T_G$ can be embedded in a well-known way into some $GL_n(\mathbf{Z})$, by making M_G and T_G act on a suitable factor ring of the group ring $\mathbf{Z}M_G$. If the groups T_G are defined in a sufficiently uniform manner, we can arrange that an isomorphism from $\hat{H}$ to $\hat{G}$ induces an isomorphism from $(H\,]\,T_H)\hat{}$ to $(G\,]\,T_G)\hat{}$ sending $\widehat{T_H}$ to $\widehat{T_G}$. In this situation it is not hard to deduce that $G \sim_{GL_n} H$ in $GL_n(\mathbf{Z})$. A second application of Theorem 2 completes the proof.

Our construction of semisimple splittings differs from those in the literature. Roughly speaking, we construct a certain canonical family $\mathcal{X}(G)$ of nilpotent supplements for N_G in G, G being any polycyclic group. To get the required "uniformity", we then choose $C_G/Q_G \in \mathcal{X}(G/Q_G)$ simultaneously for all $G \in \mathcal{C}$, using the isomorphisms between the groups G/Q_G. The group T_G is defined to act trivially on C_G and to act like the Jordan semisimple component of $\mathrm{Inn}(C_G)|_{N_G}$ on N_G; the existence of such a T_G is a direct consequence of the fact that C_G is nilpotent.

OUTLINE PROOF OF THEOREM 2. Consider an algebraic $\mathbf{Q}$-group G and a set $\mathcal{C}$ of soluble-by-finite subgroups of $G(\mathbf{Z})$, contained in a single $\sim_G$-class. Using Theorem 4 and induction on derived length, we may assume that $\mathcal{C}$ consists of abelian groups. Now consider some special cases. If $\mathcal{C}$ consists of unipotent groups, the result is an easy consequence of Theorem 5. If $\mathcal{C}$ consists of abelian d-groups and $G = GL_n$, the result is proved with the help of Theorem 6: suppose $X \sim_G Y$ in $GL_n(\mathbf{Z})$; we diagonalize X and Y over some ring $\mathfrak{o}$ of algebraic integers, and then for each prime $\mathfrak{p}$ of $\mathfrak{o}$ we find a permutation matrix $\sigma(\mathfrak{p})$ such that $Y \equiv X^{\sigma(\mathfrak{p})} \bmod \mathfrak{p}$. There exists a permutation matrix τ such that $\{\mathfrak{p} \mid \sigma(\mathfrak{p}) = \tau\}$ is ample, and one can deduce that then $Y = X^\tau$. To deduce the result for abelian d-subgroups in a general G one uses the conjugacy of maximal tori.

A major part of the proof consists in reducing the problem to the special cases mentioned; this involves a series of rather complicated arguments which we cannot go into here. At several points in the proof, and particularly in the proof of Theorem 5, a key role is played by finiteness theorems of Borel [B] and Borel-Serre [BS].

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