book; but one could hope to intoduce each of them and include at least one nontrivial result for each.

Proof Theory has largely gone its own way, and can be treated briefly in a general logic text. Most important is the Gentzen method of elimination of cuts, which should be discussed at least briefly.

Finally, there are subjects which cut across these fields. Most notable is the theory of admissible sets, which is both a unifying idea and a valuable tool. One would hope to include an introduction to this topic.

How well does Manin's book cover this material? The basic material on first-order logic is there and is done well. The only result in Model Theory is the Löwenheim-Skolem Theorem. In Recursion Theory, he covers the material mentioned above through the arithmetical hierarchy, but with some essential results (like the Enumeration Theorem for partial recursive functions) missing. In set theory, constructible sets and forcing (in the Boolean model form) are treated fairly extensively, but the other advanced topics are not mentioned. There is no Proof Theory and no mention of admissible sets.

On the other hand, Manin has proved several significant results not in the above list, e.g., Higman's Theorem on embeddings in finitely presented groups and the Kochen-Specker results on quantum logics. There is no doubt that these results and their proofs are interesting; but the techniques are special to the problem and not of much general use.

After this long discussion on content, a word about style. The book meets reasonable standards of clarity and elegance; but it's outstanding feature is its liveliness. Manin is interested in everything; and there are many (perhaps too many) asides on topics connected in some way with logic. No one should be bored by this book.

A lively book treating the fundamentals of logic and some important advanced topics–surely this is enough to make the book worthwhile. Still, I cannot help looking forward to the book which will treat most of the topics above in an equally lively way. It will surely be the logic textbook for the 1980s.

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Saks spaces and applications to functional analysis, by J. B. Cooper, Mathematics Studies, Vol. 28, North-Holland, Amsterdam, New York, Oxford, 1978, x + 325 pp.

The term "Saks space" has no fixed meaning in the literature, but it always refers to some variant of the following situation: a normed vector space $(E, \| \ \|)$ provided with a distance function or a topology τ. Depending on the author, a Saks space is either a triple $(E, \| \ \|, \tau)$ or a pair (B, τ_B) where B is the unit ball $\{x \in E: \|x\| \leqslant 1\}$ and τ_B is the induced metric or topology. The conditions imposed on E, $\| \ \|$ and τ may vary; usually it is assumed that

(*) τ is coarser on E than the norm topology and B is closed in (E, τ), together with further restrictions like metric completeness of (B, τ_B) or local convexity of (E, τ). The term "two-norm space" is used when τ is a norm on E coarser than $\| \ \|$. A typical example is the space L^∞ of bounded measurable functions on [0, 1] provided with two norms: the norm $\|f\|_\infty = \operatorname{essup}|f|$ and the norm $\|f\|_1 = \int|f|\, d\mu$ (or with a distance function for the convergence in measure, say).

In the theory of Saks spaces, besides the usual convergence with respect to either $\| \ \|$ or τ, we have a new notion, called the *γ-convergence*: it means the convergence with respect to τ together with boundedness with respect to $\| \ \|$. Suitable morphisms are γ-continuous linear operators, i.e., linear operators which are

(**) τ-continuous on $\| \ \|$-bounded sets.

The term "Saks space" was introduced by W. Orlicz [4] in 1950 to honor Stanislaw Saks, a Polish mathematician who inspired the general theory by his paper [5] published in 1933 in the Transactions. Saks was a significant contributor to measure theory and functional analysis, the author of the classical *Theory of the integral.* Nowadays his name is connected with the Vitali-Hahn-Saks theorem and the Banach-Saks property. Few people know of his role in promoting the Baire category method in functional analysis: as the referee of the paper [1] by S. Banach and H. Steinhaus he suggested to replace the original laborious constructive proof by a simple argument, which later became a standard way of proving their celebrated result. Saks was killed by Gestapo in 1942, at the age of 44.

The initial work on Saks spaces (W. Orlicz, A. Alexiewicz) was concerned with the problem of γ-continuity of pointwise limits of linear γ-continuous operators in case (B, τ_B) is metric complete; the Baire category theorem for (B, τ_B) together with a special technique due to Saks [5] (based on the so-called condition Σ_1) was the main tool here, leading to results which could not be obtained by previously known methods of functional analysis (e.g., to the Mazur-Orlicz theorem on compatibility of summability methods for bounded sequences).

In many problems of analysis γ-continuity is just what is needed when the assumption of τ-continuity is too strong and $\| \ \|$-continuity is not enough. For instance, let $C^\infty(X)$ be the space of bounded complex-valued continuous functions on a locally compact space X. If one looks for a class of linear functionals on $C^\infty(X)$ corresponding to finite Borel measures on X, then the continuity with respect to the sup norm $\| \ \|$ is not sufficient (it actually yields a much bigger class of measures on the Stone-Čech compactification βX), while the continuity with respect to the topology τ_K of uniform convergence on compact subsets of X requires that the measure be of compact support. By a significant theorem of Grothendieck, *if condition* (*) *is satisfied and* τ *is locally convex, then the set of linear γ-continuous functionals on E is the norm closure of the set of τ-linear functionals in the space conjugate to* $(E, \| \ \|)$. (This theorem was proved in [3] under an additional assumption that B is τ-precompact; in [2, p. 91], it was stated as "théorème de Grothendieck" without this superfluous assumption.) Applying the theorem to the case $E = C^\infty(X)$, one gets an immediate corollary: a linear functional on $C^\infty(X)$ is the integral with respect to a complex-valued measure if and only if it is γ-continuous.

Another reason for dealing with Saks spaces is that (B, τ_B) may have nicer topological properties than $(E, \| \|)$ and (E, τ). For instance, if E is the space H^∞ of bounded analytic functions in the unit disk with the sup norm $\| \|$ and the topology τ of uniform convergence on compact sets, then (B, τ_B) is compact whereas neither of the spaces $(E, \| \|)$, (E, τ) is locally compact.

A. Wiweger constructed, under some mild assumptions about $(E, \| \|, \tau)$, a locally convex topology $\gamma = \gamma(\| \|, \tau)$ on E, called the *mixed topology*, which is the finest linear topology coinciding with τ on $\| \|$-bounded sets and, at the same time, the coarsest linear topology such that linear operators $E \to E_1$ are continuous with respect to γ if and only if they are γ-continuous in the previous sense (**). This result marked a shift of the theory of Saks spaces towards that of linear topological spaces.

"The author feels that the theory of Saks spaces is sufficiently well developed and useful to justify an attempt at a first synthesis of the theory and its applications"–is stated in the preface to the book under review. The usual dull preliminary part is successfully omitted and the book begins with an exposition of the theory of mixed topologies in a general setting. One might think that a suitable generalization of $(E, \| \|, \tau)$ is E provided with two topologies. The author, however, rightly assumes that for the first of the two topologies only the bounded sets are relevant and therefore considers a vector space E provided with a bornology $\mathfrak{B}$ (that is, an axiomatically given family of sets called bounded) and a topology τ (there is a concept of a bitopological space in the literature, but it has very little in common with Saks spaces). The main object of interest is here a space E with two structures $\mathfrak{B}$ and τ rather than E with the single mixed topology $\gamma(\mathfrak{B}, \tau)$; in particular, all canonical constructions (subspaces, products etc.) are carried out so that this double structure be preserved. In this setting e.g. a Saks space $(E, \| \|, \tau)$ is γ-complete if and only if it is the Saks-space projective limit of a system of Banach spaces (in contrast to the locally-convex-space projective limit, which is something else).

There is a long list of examples: duals of Fréchet spaces, various l_p-spaces, the space $C^\infty(X)$, the space $C_{00}(X)$ of functions in $C^\infty(X)$ with compact support, $H^\infty(G)$, C^*-algebras, the space $L(E, F)$ of continuous linear operators, direct product $\prod E_n$ and direct sum $\sum E_n$ of Banach spaces, spaces of Lipschitz functions–each with suitable family $\mathfrak{B}$ of bounded sets and a topology τ. One might add, however, another example: the algebra $L_1(G)$ for a commutative locally compact group G with the L_1-norm $\| \|_1$ and the spectral norm $\| \|_s$ (at a seminar in 1962 in Seattle J. M. G. Fell showed that condition (*) is satisfied in this case).

Four successive chapters (more than a half of the book) are devoted to special classes of Saks spaces. One can learn about mixed topologies on $C^\infty(X)$ for a completely regular X, on $L^\infty(\mu)$ for a positive Radon measure μ on a locally compact space, on von Neumann algebras, and on $H^\infty(G)$ for an open domain $G \subset \mathbf{C}$. The author's major concern is to study the mutual relations between mixed topologies and strict topologies on $C^\infty(X)$ and on $H^\infty(G)$, introduced by R. C. Buck and studied later at various levels of generality. There were, in fact, two topics developed independently: Wiweger's general construction (1957), J. Mařík's topologization of $C^\infty(X)$ (1957),

D. J. H. Garling's inductive-limit approach to mixed topology (1964) and some other results were in one line of papers referring to their predecessors while Buck (1952, 1958) and his followers formed the other. It took over 10 years till it was realized that in both cases it was the same topology. J. B. Cooper's book combines results and terminologies of both streams of research.

Saks-space methods may be used to get new proofs of known results or to shed new light on them. In the book we find, e.g., a functional-analytic proof of Prohorov's theorem on the existence of projective limits of measures on projective spectra of completely regular spaces, a proof that $C(K)$ has the Dunford-Pettis property, and a proof of Kaplansky's density theorem. In his exposition of von Neumann algebras the author uses three natural mixed topologies β_σ, β_s and β_{s*} on $L(H)$ rather than the ultraweak and ultrastrong topologies; the main result here is that β_{s*} is a Mackey topology. There are also Saks-space generalizations of topics known in the Banach-space case, like the commutative part of the Gelfand-Naĭmark theorem and the isometry between $C(X) \hat{\otimes} E$ and $C(X, E)$.

A categorical approach to some results on Saks spaces is presented in an appendix. The main stress is laid on functorial dualities of various degrees of sophistication. Besides the known duality between Banach spaces and compact Saks spaces, one can learn about the dualities between regular compactological spaces and commutative Saks C^*-algebras with unit, between compactological groups, monoids or semigroups and certain Saks C^*-coalgebras (with applications to Bohr compactification). There is also a description of some completion process for categories which includes, as a special case, the transition from Banach spaces to Saks spaces.

The author has attempted to make the core of the book as self-contained as possible so that it will be useful to nonspecialists. It is assumed that the reader is familiar with rudiments of respective topics, but all nonstandard terms are defined and auxiliary results are explicitly stated (though often without proofs). The style of exposition varies: basic facts are presented at a text-book pace, more special results and comments–in the form of lecture notes or a survey article. Some ideas are only outlined. Each chapter is augmented with notes (brief historical remarks and references to original papers) and the bibliography. The references (either explicitly quoted or just added for the sake of completeness) appear to be very carefully collected; yet the author–similarly as his predecessors–fails to credit Grothendieck ([3] or [2]) for the result mentioned above.

The book ends with an epilogue where open problems and possible further development are discussed.

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