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Innovation processes, by Yuriy A. Rozanov, John Wiley & Sons, New York, Toronto, London, and Sydney, 1977, vii + 136 pp., $14.50.

Complex valued random variables ξ_s, $s \in I$, with finite second moments being simply L_2-functions on a probability space, problems involving only the second moments are naturally set in the corresponding Hilbert space, the expectation $E[\xi_s\bar{\xi}_t] \equiv B(s, t)$ serving as the inner product. Throughout this review all random variables will be assumed to have second moments, and also, for convenience only, the first moment will be taken to be zero. The parameter set I will be taken to be the interval (α, β) on the line, where α or β may be infinite. $B(s, t)$ is the covariance function. Then $(\xi_s, s \in I)$ is a stochastic process; or a curve in Hilbert space. For $t \in I$, let $H_t(\xi)$ be the closed linear hull of $\{\xi_s: \alpha < s \leqslant t\}$, let $H(\xi)$ be the closure of the union of the $H_t(\xi)$, $t \in I$, and let P_t be the operator of orthogonal projection onto $H_t(\xi)$. The theme of Rozanov's book is the temporal evolution of the family of nondecreasing subspaces $(H_t(\xi), t \in I)$. This leads to questions in the geometry of Hilbert space naturally motivated by probabilistic considerations: $\{\xi_s: \alpha < s \leqslant t\}$ represents the observations available up to time t, and for $t < u < \beta$, $P_t\xi_u$ is the best linear predictor (in the sense of mean square error) of ξ_u in terms of the past up to time t. The process $(\xi_t, t \in I)$, will be assumed left-continuous, and this implies the same property for the family $(H_t, t \in I)$ and also the separability of $H(\xi)$. For simplicity ξ_t is taken to be complex-valued, but much recent work in the area has been devoted to vector space valued cases, and this is also the setting of Rozanov's book.

The notion of an innovations process associated with $(\xi_t, t \in I)$ is due to Cramèr [1], [2], [3]; for a related development see Hida [7]. It can be shown that there exists a finite or infinite sequence $\zeta^{(i)}$ of elements of $H(\xi)$ so that on putting $\zeta_t^{(i)} = P_t\zeta^{(i)}$ the following conditions hold: (i) $H_t(\zeta^{(i)}) \perp H_t(\zeta^{(j)})$ for $i \neq j$; (ii) setting $F_t^{(i)} = E[|\zeta_t^{(i)}|^2]$, $F^{(j)}$ is absolutely continuous with respect to $F^{(i)}$ for $j > i$; $H_t(\xi) = \Sigma_i \oplus H_t(\zeta_i)$. Of course each $(\zeta_t^{(i)}, t \in I)$ is a process with orthogonal increments. The length of the sequence $\zeta^{(i)}$ is called

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the *multiplicity* M. So $1 \leqslant i < M + 1$, with M a positive integer or ∞. Now $(\zeta_t^{(i)}, t \in I, 1 \leqslant i < M + 1)$ is an *innovation process*. It leads directly to a *canonical representation* for ξ, namely

$$\xi_t = \sum_i \int_0^t c_i(t, u)\, d\xi_u^{(i)}$$

where the functions c_i satisfy

$$\sum_i \int_0^t |c_i(t, u)|^2\, dF_i(u) < \infty.$$

The innovations process is not uniquely determined by $(\xi_t, t \in I)$, but the multiplicity M is, and so is the sequence $[F^{(i)}]$, $1 \leqslant i < M + 1$, where $[F^{(i)}]$ is the equivalence type of $F^{(i)}$ under mutual absolute continuity. The sequence $[F^{(i)}]$, $1 \leqslant i < M + 1$, is called the *structural type* of $(\xi_t, t \in I)$. It is in fact uniquely determined by the family $(H_t(\xi), t \in I)$, and, conversely it determines this family up to isometry: another such family $(H_t(\eta), t \in I)$ will have the same structure type if and only if there is an isometry from $H(\eta)$ onto $H(\xi)$ taking $H_t(\eta)$ onto $H_t(\xi)$ for each $t \in I$.

Using ideas from the theory of Gohberg and Kreĭn [6] on Volterra operators it is possible to show that a sufficient condition for $(H_t(\xi), t \in I)$ and $(H_t(\eta), \eta \in I)$ to have the same structural type is the following:

(*) there exists a linear homeomorphism A between $H(\eta)$ and $H(\xi)$ such that $A\eta_t = \xi_t$ for all $t \in I$, and $(I - A^*A)$ is a Hilbert-Schmidt operator.

The condition (*) was originally introduced by Feldman [5] in a slightly different context. He showed that if the processes $(\xi_t, t \in I)$ and $(\eta_t, t \in I)$ are Gaussian, so that each has a distribution entirely determined by its covariance, then these distributions are mutually absolutely continuous if (*) holds, and they are orthogonal if (*) fails.

Of course given two processes $(\xi_t, t \in I)$ and $(\eta_t, t \in I)$ with covariances $B_\xi(s, t)$, $B_\eta(s, t)$ one would like criteria in terms of B_ξ and B_η for deciding whether or not the relation (*) holds. A good deal of work has been done on this problem. For example, Shepp [9] characterized all B_η so that (*) holds relative to $B_\xi(s, t) = \min(s, t)$, $I = [0, 1]$ (this is the covariance of the Wiener process).

The notions introduced by Cramèr are significant generalizations of concepts known from the stationary case, that is the case $I = (-\infty, \infty)$, $B(s, t) = B(t - s)$. This is the situation investigated in the classical studies on prediction theory by Kolmogorov, Wiener, Wold and others. For an exposition and references consult the relevant chapter of Doob [4], including the pertinent historical notes at the end of the book; a more recent account, discussing the case ξ_t is n-dimensional, is given in Rozanov [8]. It can happen that the spaces H_t are all equal to $H_{-\infty}(\xi) = \bigcap_{t \in I} H_t(\xi)$ and this is known as the *deterministic* case; otherwise one is in the *nondeterministic* case, a subcase of which is the *purely nondeterministic* situation $H_{-\infty}(\xi) = \{0\}$. (This is Cramèr's terminology; various mutually inconsistent terminologies are in use.) The problem of characterizing the different cases analytically in terms of $B(t)$ was solved by Kolmogorov. Since B is positive definite it is the Fourier-Stieltjes transform of a distribution function, the *spectral distribution function*, and it is in terms of this that characterizations are given. In any case, Hanner, and Karhunen had shown that, (in Cramèr's terminology) if ξ is a (scalar) stationary process which is purely nondeterministic then it has multiplicity $M = 1$ and spectral type $([m])$, where m is Lebesgue measure. This contrasts with the results for nonstationary processes where, even in the purely nondeterministic case any value of M can occur.

In the book under review Rozanov surveys the indicated problem area, including the situation where ξ_t may be vector space valued. Rozanov himself has made many contributions toward the solutions of these problems. It seems remarkable that he manages to give complete proofs and numerous examples in this book of 133 short pages. The translation from the Russian, edited by A. V. Balakrishnan, reads very well. The book should be welcome by both novice and experts in the field.

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Equations of mixed type, by M. M. Smirnov, Transl. Math. Monographs, Vol. 51, American Mathematical Society, Providence, R. I., 1978, iii + 232 pp., $27.20.

Just what is an equation of mixed type? Equation here means partial differential equation, and if some of these are of mixed type, there must be others not of mixed type. What are they? To answer these questions we must know the labels which are attached to various classes of partial differential equations. As one would expect, the labeling process has evolved over the years in a disorderly way; by now however the terminology has stabilized for many (but far from all) classes of equations. As is the case for the problem of taxonomy in the biological sciences, the subdivision of partial differential equations into clearly defined classes has not been systematic. New terms continue to develop as the need arises. For example, the term strongly elliptic was invented to identify a special subclass of the class of elliptic equations. Classes overlap: hypoelliptic equations contain some elliptic equations and some which are not elliptic; both the class of linear equations and the class of