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GEOMETRY OF G/P

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We summarise here the main results of *Geometry of G/P-I*, . . . , IV(cf. [7], [6], [4], [5]). The purpose is to extend the Hodge-Young standard monomial theory for the group $SL(n)$ (cf. [2]) to the case of any semisimple linear algebraic group G. The problem is to give an explicit basis for the space $H^0(G/B, L)$ of sections of a line bundle L on G/B (or more generally for $H^0(X, L)$, X a Schubert subvariety of G/B), in terms of chosen nice bases for $H^0(G/B, L_i)$, where L_i are the line bundles on G/B associated to the fundamental weights. In particular, when the base field is of characteristic 0, by the Borel-Weil theorem, this problem is equivalent to giving explicit bases for all finite dimensional irreducible G-modules in terms of chosen nice bases for the fundamental representations of G. Our results provide a complete solution to this problem when G is a classical group as well as a partial answer when G is exceptional. Apart from their independent interest, our results have important applications, namely: they provide

(i) another proof of the vanishing theorem of Kempf (cf. [3])

(ii) explicit (linear) bases for the rings of invariants in classical invariant theory and

(iii) a proof of Demazure's conjecture (cf. [1]) for classical groups.

Let G be a semisimple, simply connected Chevalley group (of rank n) over a field k. Fix a maximal (split) torus T, a Borel subgroup $B \supset T$ and the corresponding roots Δ, simple roots S, etc. Let $\pi_1, \ldots, \pi_n$ be the fundamental weights. For $\pi \in X(T)$ (= the character group of T), write $\pi = \Sigma\, a_i\pi_i$, $a_i \in \mathbf{Z}$. Let $L(\pi)$ denote the line bundle on G/B so that $H^0(G/B, L(\pi)) = \{f\colon G \longrightarrow k / f(gb) = \pi(b)f(g)$, for all $b \in B$ and $g \in G\}$. Recall that π is *dominant* (*written* $\pi \geqslant 0$) iff $a_i \geqslant 0$ for all i iff $H^0(G/B, L(\pi)) \neq (0)$.

Let $Q \supseteq B$ be any parabolic subgroup of G. Let $W_Q = N_Q(T)/T$ be the Weyl group of Q and write $W = W_G$. For $\theta \in W/W_Q$, let $X(\theta)$ be the Schubert variety in G/Q associated to θ, i.e., $X(\theta) = \overline{B\theta Q}/Q$ endowed with the canonical reduced structure. We have the Bruhat (partial) order $\geqslant$ on W/W_Q, namely, $\theta_1 \geqslant \theta_2$ if $X(\theta_1) \supseteq X(\theta_2)$.

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Let $P \supset B$ be a maximal parabolic subgroup of G, associated to a fundamental weight π (or a simple root α). We say that P or π is of *classical type* if $2(\pi, \beta)/(\beta, \beta) \leqslant 2$ for all positive roots β (where (,) denotes a W-invariant inner product on $X(T)$). More generally, we say that a parabolic subgroup Q of G is of *classical type* if every maximal parabolic subgroup containing Q is of classical type. We see easily that G is a *classical group* (i.e., of type A, B, C or D) *if and only if B is of classical type.* Moreover, *any G has at least one parabolic subgroup Q which is of classical type.*

Let $[X(\theta)]$ denote the element of the Chow ring of G/P, determined by $X(\theta)$, $\theta \in W/W_P$. If $w_0 \in W$ is the element of the *largest length*, we know that $X(s_\alpha w_0)$ is the unique codimension 1 Schubert subvariety of G/P. Recall that we have

$$[X(\theta)] \cdot [X(s_\alpha w_0)] = \sum_i d_i [X(\theta_i)], \qquad d_i \in \mathbf{N},$$

where $\cdot$ denotes multiplication in the Chow ring and θ_i runs over the set of all $\theta_i \in W/W_P$ such that $X(\theta_i)$ is of codimension 1 in $X(\theta)$. Using a formula of Chevalley for d_i (cf. [1], [4]), it is seen easily that P is of classical type iff $d_i \leqslant 2$ for all i (and θ). We say that $X(\theta_i)$ is a *double divisor* in $X(\theta)$ if $d_i = 2$. A pair of elements (θ, δ) in W/W_P is called an *admissible pair* if either $\theta = \delta$ or $\theta \neq \delta$ and there exist $\{\delta_i\}$, $1 \leqslant i \leqslant s$, $\delta_i \in W/W_P$, such that

(i) $\theta = \delta_1 > \delta_2 > \cdots > \delta_s = \delta$, $X(\delta_i)$ is of codimension 1 in $X(\delta_{i-1})$, $2 \leqslant i \leqslant s$ and

(ii) $X(\delta_i)$ is a double divisor in $X(\delta_{i-1})$ for all i.

Let $W \overset{*}{/} W_P$ denote the set of all admissible pairs in W/W_P. We identify W/W_P canonically with a subset of $W \overset{*}{/} W_P$. We extend the partial order $\geqslant$ on W/W_P to an order $\geqslant$ on $W \overset{*}{/} W_P$ by defining $(\theta_1, \theta_2) \geqslant (\delta_1, \delta_2)$ if $\theta_2 \geqslant \delta_1$ (in W/W_P).

Let P_i denote the maximal parabolic subgroup of G corresponding to the fundamental weight π_i, $1 \leqslant i \leqslant n$. Write $W_i = W_{P_i}$, $1 \leqslant i \leqslant n$. Assume for simplicity of notation that $P_1, \ldots, P_r$, $r \leqslant n$, are *all* those which are of classical type. (Viz., $r = n$ if G is a classical group, $r = 5$ if G is of type E_6, etc.) Let $Q = \bigcap_{i=1}^r P_i$. Fix an $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbf{Z}^+)^r$. By a *Young diagram of type* $\mathbf{a}$ *or multidegree* $\mathbf{a}$, we mean a pair (θ, δ) defined as follows:

$$\theta = (\theta_{ij}), \quad \delta = (\delta_{ij}), \quad (\theta_{ij}, \delta_{ij}) \in W \overset{*}{/} W_P$$

for $1 \leqslant j \leqslant a_i$ and $1 \leqslant i \leqslant r$. We say that a Young diagram (θ, δ) is *standard on* $X(w)$ (or *relative to*) $w \in W/W_Q$ (and *written* $w \geqslant (\theta, \delta)$), if there exists a pair (α, β), called a defining *pair for* (θ, δ), defined as follows. For i, j as above,

(i) $\alpha = (\alpha_{ij})$, $\beta = (\beta_{ij})$, $\alpha_{ij}, \beta_{ij} \in W/W_Q$,

(ii) each α_{ij} (resp. β_{ij}) is a *lift* for θ_{ij} (resp. δ_{ij}) and

(iii) $w \geqslant \alpha_{11} \geqslant \beta_{11} \geqslant \cdots \geqslant \alpha_{ij} \geqslant \beta_{ij} \geqslant \alpha_{ij+1} \geqslant \beta_{ij+1} \geqslant \cdots \geqslant \alpha_{ia_i} \geqslant \beta_{ia_i} \geqslant \alpha_{i+11} \geqslant \beta_{i+11} \geqslant \cdots \geqslant \alpha_{ra_r} \geqslant \beta_{ra_r}$ (in W/W_Q).

(If $a_t = 0$ for *any* t, $1 \leqslant t \leqslant r$, the family $\theta_{t-}, \delta_{t-}, \alpha_{t-}, \beta_{t-}$ is understood to be empty.)

Main results. Let $L_i = L(\pi_i)$ be the *ample* generator of Pic G/P_i, $1 \leqslant i \leqslant n$. Then we have the following:

THEOREM 1. *There exists a basis* $\{p(\theta_1, \theta_2)\}$ *of* $H^0(G/P_i, L_i)$, *parametrised by* $(\theta_1, \theta_2) \in \overset{*}{W}/W_P$, $1 \leqslant i \leqslant r$ $(\leqslant n)$, *such that*

(1) $p(\theta_1, \theta_2)$ *is a weight vector of weight* $\chi(\theta_1, \theta_2) = -\frac{1}{2}(\theta_1(\pi_i) + \theta_2(\pi_i))$,

(2) *for each* $w \in W/W_i$, *the restriction of* $p(\theta_1, \theta_2)$ *to* $X(w)$ *is not zero iff* $w \geqslant \theta_1$ *(in* W/W_i*) and*

(3) $\{p(\theta_1, \theta_2)/w \geqslant \theta_1\}$ *is a basis of* $H^0(X(w), L_i)$.

Let $\pi = \Sigma_{i \leqslant r} a_i \pi_i$, $a_i \in \mathbf{Z}^+$. To each Young diagram (θ, δ) of multidegree $\mathbf{a} = (a_1, \ldots, a_r)$, we define the monomial $p(\theta, \delta) \in H^0(G/Q, L(\pi))$ as follows:

$$p(\theta, \delta) = \prod_{1 \leqslant i \leqslant r} \prod_{1 \leqslant j \leqslant a_i} p(\theta_{ij}, \delta_{ij}).$$

Note that $p(\theta, \delta)$ is a weight vector of weight

$$\chi(\theta, \delta) = \sum_{1 \leqslant i \leqslant r} \sum_{1 \leqslant j \leqslant a_i} -\tfrac{1}{2}(\theta_{ij}(\pi_i) + \delta_{ij}(\pi_i)).$$

We say that the monomial $p(\theta, \delta)$ is *standard on* $X(w)$ (or *relative to*) $w \in W/W_Q$ if (θ, δ) is standard on $X(w)$, i.e., $w \geqslant (\theta, \delta)$.

THEOREM 2. *Let* $Q = \bigcap_{i \leqslant r} P_i$ *be of classical type and* $\pi = \Sigma_{i \leqslant r} a_i \pi_i$ *be dominant. (For example,* $Q = B$ *if* G *is of type* A, B, C *or* D.) *Then for every* $w \in W/W_Q$, *we have*:

(1) *Standard monomials* $p(\theta, \delta)$ *on* $X(w)$ *of multidegree* $\mathbf{a} = (a_1, \ldots, a_r)$ *form a basis of* $H^0(X(w), L(\pi))$. *Consequently we have the following*:

(a) *Character formula*: $\operatorname{char}[H^0(X(w), L(\pi))] = \Sigma_{w \geqslant (\theta, \delta)} \exp \chi(\theta, \delta)$.

(b) *For any dominant* $\pi' = \Sigma_{i \leqslant r} a'_i \pi_i$, *the natural map*

$$H^0(X(w), L(\pi)) \otimes H^0(X(w), L(\pi')) \longrightarrow H^0(X(w), L(\pi + \pi'))$$

is surjective.

(c) *The natural restriction map*

$$H^0(G/Q, L(\pi)) \longrightarrow H^0(X(w), L(\pi))$$

is surjective. In other words, if $L(\pi)$ *is ample (i.e.,* $a_i > 0$ *for all* i), *then* $X(w)$ *is projectively normal for the imbedding given by* $L(\pi)$.

(2) $H^j(X(w), L(\pi)) = (0)$ *for* $j \geqslant 1$.

REMARK. Theorems 1 and 2 above remain valid when the base ring is $\mathbf{Z}$ instead of a field k.

Consequences of the main results.

1. *Ideal theory of Schubert varieties.* One of the most important consequences of standard monomial theory is that it provides a good hold of the ideal theory of Schubert varieties (viz., of their unions, intersections and hyperplane sections, etc.). This allows us to deduce the following:

2. *Vanishing theorem* (cf. [3], [4]). *Let Q be of classical type and $\pi \geqslant 0$ relative to Q. Then for every Q-stable Schubert variety $X(w)$ (in particular $X(w) = G/B$) in G/B, $H^j(X(w), L(\pi)) = (0)$ for $j \geqslant 1$.* (Kempf proved this for the case when Q is maximal parabolic and *quasi-minuscule*.)

3. *Invariant theory and Determinantal varieties* (cf. [6]). It can be shown that the results of De Concini and Procesi giving a basis of the ring of invariants in *classical invariant theory* by means of certain "*double standard tableaux*" are particular cases of Theorem 2; in fact, Theorem 2 allows us to identify the varieties defined by the above rings of invariants with suitable affine open subsets ("determinantal varieties") of certain Schubert varieties in G/P (G a classical group and P a maximal parabolic subgroup of G). These results provided, however, a motivation for our standard monomial theory.

4. *Demazure's conjecture* (cf. [1, p. 83]). It can be shown that this conjecture is a consequence of the statement (1) (c) of Theorem 2. Hence this conjecture holds when G is of type A, B, C or D.

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