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During the thirty years which have elapsed since the publication in *Mathematics Magazine* of a justly famous paper by M. H. Stone [12], the Stone-Weierstrass theorem, as the contents of this article have come to be collectively known, has exercised a pervasive influence on the development of modern analysis. Aside from the effect of Stone's pioneering work with what can be termed an algebraic approach to analysis, the theorem itself has direct applications in areas ranging from spectral theory to group representations. Subsequent generalizations, moreover, have served to extend the power and utility of Stone's theorem. One striking example is the well-known extension due to Errett Bishop [2]; obtained in response to a question raised by Šilov [11], Bishop's result has some of its deepest implications in the theory of uniform algebras (e.g., see Burckel's monograph [4]).

Stone's generalization of the Weierstrass approximation theorem is, of course, primarily an assertion about the (real or complex) algebra $C(X)$ of all scalar valued continuous functions on a compact Hausdorff space X, where $C(X)$ comes equipped with the topology of uniform convergence on X. From the very outset, however, further generalizations which relax the conditions on X, or otherwise broaden the domain of application to include this or that topological algebra of scalar or vector valued continuous functions, have been of interest. As noted by Stone himself (op. cit.), if X is locally compact, then utilization of the one-point compactification of X yields a variation phrased in terms of the subalgebra $C_0(X)$ consisting of those $f \in C(X)$ which vanish at infinity, where the topology on $C_0(X)$ is still that of uniform convergence. On the other hand, even if X is only assumed to be, say, a completely regular Hausdorff space, endowing $C(X)$ with the compact-open topology also leads directly to an instance (cf. [9]). Then there is R. C. Buck's version for the strict topology on the subalgebra $C_b(X)$ of all bounded continuous functions, where X is again taken to be locally compact [3], and the list could go on. But rather than considering each individual case as the need arises, or the whim may strike, it becomes natural to ask whether there isn't some way to unify this growing body of results and put it, so to speak, all under one roof.

This question has been treated with a good deal of success by Leopoldo Nachbin [7], [8] through the use of an idea embedded in yet another line of investigation with roots extending back to the classical Weierstrass approximation theorem–namely, that stemming from the Bernstein approximation problem. Posed by S. Bernstein [1] in connection with his work relating to Hadamard's famous query concerning the classification of quasianalytic functions, the problem, roughly speaking, is to obtain polynomial approximations of (certain) continuous functions on the entire real axis via weighted analogues of the supremum norm. A key to the formulation was Bernstein's use of weights to both select the (vector) space of functions to be approximated and, at the same time, describe the measure of approximation; it is this point on which Nachbin has based his development.

To briefly set the context introduced by Nachbin (op. cit.), consider a set V of nonnegative upper semicontinuous functions (called weights) defined on a topological space X which is normally assumed to be at least a completely regular Hausdorff space. A (weighted) locally convex space $CV_\infty(X)$ is associated with the pair (X, V) by taking the vector space consisting of all $f \in C(X)$ such that, for each $v \in V$, fv vanishes at infinity on X, and then equipping it with the weighted topology ω_V generated by the seminorms p_v, one for each $v \in V$, defined by putting $p_v(f) = \|fv\|_\infty$. (All of the examples mentioned above, by the way, as well as numerous other familiar spaces of continuous functions, can be realized in the form $CV_\infty(X)$ through appropriate choices of the pair (X, V).) Next, fixing a subalgebra A of $C(X)$, let W be a vector subspace of $CV_\infty(X)$ which is, in addition, a module over A; Nachbin's "weighted approximation problem" is to describe the closure of W in $CV_\infty(X)$ under these circumstances. More precisely, Nachbin was interested in determining when the closure of W would coincide with the closed vector subspace consisting of all $f \in CV_\infty(X)$ for which, given $\varepsilon > 0$, $v \in V$, and $x \in X$, there exists some $w \in W$ such that

$$\sup\{|f(y) - w(y)|v(y): y \in [x]_A\} < \varepsilon,$$

where $[x]_A = \{y \in X: g(y) = g(x), g \in A\}$. Referring to W as being *localizable under A* whenever this happens to be the case, Nachbin proceeded to establish a sequence of successively less demanding conditions on the triple (X, V, A) which nonetheless would guarantee localizability, and the resulting generalizations of the Stone-Weierstrass theorem go a long way toward achieving the desired unification. The most accessible of these, perhaps, is the one arising in the so-called bounded case of the weighted approximation problem.

THEOREM (NACHBIN [8]). *Assuming A to be selfadjoint in the complex case, if each $g \in A$ is bounded on the support of every $v \in V$, then W will always be localizable under A.*

A detailed discussion of Nachbin's work with the bounded case and its relatives can be found in his monograph on weighted approximation [9]. In the interim, however, additional progress has been realized on a number of fronts: Nachbin's treatment of localization, for instance, has been broadened to encompass Bishop's generalization of the Stone-Weierstrass theorem;

many results originally proved for $CV_\infty(X)$ have been rephrased for its vector valued analogue $CV_\infty(X; E)$, or sometimes even in terms of much more exotic settings; and considerable information has come to light concerning the locally convex structure of weighted spaces. The book under review primarily serves to chronicle the contributions made by the author and others belonging to a group of Nachbin's first and second generation mathematical descendants. In many respects, this book can be regarded as an expanded and updated edition of [9].

The heart of the present book is contained in Chapter 5. Here, by utilizing closed partitions of X which may be coarser than that formed by $\{[x]_A : x \in X\}$, the author reconstructs the bounded case and Nachbin's other localization results without having to require that A be selfadjoint and where W is now an A-module in $CV_\infty(X; E)$. In addition, Prolla reaches a (formally?) stronger conclusion which he terms 'sharp localizability' and which in the classical bounded case noted above, for example, translates into the following: for any $f \in CV_\infty(X)$ and any $v \in V$, there is some $x \in X$ such that $\inf\{p_v(f - g): g \in W\} = \inf\{\sup\{|f(y) - g(y)|v(y): y \in [x]_A\}: g \in W\}$. As with Nachbin's development in [9], all of the various sufficient conditions for (sharp) localizability are essentially derived from one basic result (Theorem 5.11, p. 84); indeed, most of the book is devoted to applications, special cases, and variations of this theorem.

Beginning in Chapter 1 with a rather detailed look at the situation as it pertains to $C(X; E)$, the reader is given a preview, proofs and all, of the material which is later to be developed in much more generality; the results of Chapter 5 have also been specialized to the cases of $C_0(X; E)$ (Chapter 6) and $C_b(X; E)$ under the strict topology (Chapter 7). Among the applications, a weighted version of Dieudonné's well-known density theorem for tensor products (Chapter 2) and Tietze-Urysohn type extensions (Chapter 3) have been singled out for particular attention. Polynomial algebras (Chapter 4) and the approximation property in weighted spaces (Chapter 8) are discussed, while Chapter 9, the last and longest, deals with weighted approximation in a nonarchimedean setting.

As I previously hinted, a large portion of the book is devoted to either enlarging upon or reproducing various papers by the authors and some of his colleagues, and the presentation has not emerged unscathed from the pitfalls inherent in such an approach to exposition. For one thing, despite the author's statements in his Preface which tend to conjure up visions of something akin to a textbook, claims like "the book is largely self-contained" and "the book can be used by graduate students who have taken the usual first-year real and complex analysis courses", one encounters such as the following (p. 89):

"PROOF Define γ on $\mathbf{R}$ as in the proof of Theorem 9, [46], and then apply Theorem 5.17, above."

And this is to establish one of the centerpieces–the quasianalytic criterion for sharp localizability! (Except for renumbering, it exactly reproduces the proof given in [6].) Not even the proof of the main theorem (5.11) escapes dependence on outside sources, while lesser lights, like Theorem 7.17 (p. 133), often receive even shorter shrift–viz.,

"PROOF: See Fontenot [24]."

Nor has much effort been wasted in trying to avoid the redundancies and inconsistencies which tend to be introduced by this splicing process. There are symbols which change meaning–e.g., $K^+(X)$ surely has a different interpretation on p. 95 than the one given on p. 92; and there are meanings which change symbols–the notation for $[x]_A$ as introduced in Chapter 1 undergoes an unannounced transformation in Chapter 5 (to match that of the articles there being transcribed). Some notation and terminology, $C(X; [0, 1])$ on p. 40, for instance, or 'spherically complete' (p. 183), comes unencumbered by any sort of definition, while the familiar notion of 'vanishing at infinity' is blessed with two (p. 79 and p. 90), which, incidentally, don't agree. Then too, results have a way of cropping up again without so much as a cross reference. A few examples follow: Theorem 5.47 (p. 105) is an obvious special case of Corollary 5.22 (p. 90); Corollary 5.36 (p. 95) is repeated on p. 131 as Theorem 7.14(4); Theorem 9.9 (p. 156) is reincarnated as Theorem 9.38 (p. 178); and et cetera.

These lapses in craftsmanship are, in my opinion, symptomatic of a more serious flaw. Whether intentional on the part of the author, I cannot say, but efforts to disguise the anthological character of the book have tended to create the (false) impression that here one has the whole story. The discussion of completeness and duality which occurs in §§4 and 5 of Chapter 5 provides one case in point: except for a few paragraphs at the end of §5, this material is, for all practical purposes, a verbatim copy of [10], a paper by the author which, interestingly enough, hasn't found its way into the bibliography; on the other hand, the relevance of the presentation may have come into question if mention had been made of deeper results like those obtained, albeit in the scalar case, by A. Goullet de Rugy [5]. (But then, reference to related literature is sporadic throughout.) While [10] doesn't appear to be one of them, there are references, by the way, which don't show up in the bibliography; some are embedded in the text, but the scheme escapes me.

Repeated generalization and specialization of the same results can quickly become boring, and the pattern established in the first chapter recurs tediously many times before the book has run its course. There can be no doubt, however, that the author has significantly contributed to the theory instigated by his mentor even if this in itself is not sufficient to sustain a book on the subject. Moreover, the weighted approximation problem (or, in the author's terminology, the Bernstein-Nachbin approximation problem) does provide a viable context in which to consider the Stone-Weierstrass theorem and its many variants; the attendant weighted spaces $CV_\infty(X; E)$, called Nachbin spaces by the author, also form a framework within which a wide range of other topics can be systematically and simultaneously studied over entire spectra of spaces of continuous functions; and a number of interesting questions yet remain to be resolved. All in all, it is indeed lamentable that the method used in constructing this book was adopted in preference to a broader, more forthright, treatment.

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