Faden goes on to apply these tools to general problems of constrained and unconstrained optimization. He presents some results on existence and uniqueness of optimal solutions, and a number of variations on "shadow-price" conditions to characterize constrained optima.

The analytical tools described and developed in the first third of the book are used to analyze various problems of spatial economics in the latter two thirds of the book. There are discussions of the real estate market, the transportation and transhipment problems, several variants on the von Thünen system mentioned earlier in this review, several models of industrial location, and discussions of a number of other topics in spatial economics. These models will no doubt stimulate some fruitful interaction between measure theorists and spatial economists; perhaps the intersection of these two disciplines may yet be of positive measure!

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Obstruction theory on homotopy classification of maps, by Hans J. Baues, Lecture Notes in Math., vol. 628, Springer-Verlag, Berlin, Heidelberg, New York, 1977, xi + 387 pp.

In the beginning was the word, and the word was homology. The great breakthrough in the successful attempt to apply algebraic methods in topology was the discovery of the homology groups and the proof of their topological invariance. Before the homology groups were explicitly described we had the idea of the homology class of a cycle on a manifold or, more generally, a polyhedron, but the description of the collection of homology classes was arithmetical rather than algebraic. That is to say, the great pioneers spoke of Betti numbers and torsion coefficients. It is generally supposed that Emmy Noether was responsible for observing that in fact the homology classes of cycles form an abelian group and that the Betti numbers and torsion coefficients were simply the invariants of finitely generated homology groups. The topological invariance of the homology groups is a truly wonderful result. We define these groups in terms of a very specific and arbitrary combinatorial structure on the topological space and then prove that they are in fact independent of that structure.

Thus algebraic topology was born. Subsequently came the cohomology groups. At first the view was taken that the combinatorial structure on the space gave rise to chain groups and that there were two operators on these chain groups, the boundary operator, or lower boundary operator as it was sometimes called, and the coboundary operator, or upper boundary operator as it was sometimes called. Subsequently it was realized that this was not a good point of view. One should exploit the natural duality to introduce not only chain groups but cochain groups and then one obtained cohomology groups from the cochain groups by a method entirely analogous to that whereby one obtained homology groups from the chain groups. This point of view, so obvious to us today, took a long time to establish itself because the early pioneers tended to think exclusively in geometric terms. It was thus very easy for them to imagine a chain on a manifold, which they could think of as being made up out of combinatorial pieces of the manifold, taken with certain multiplicities; but it was difficult for them to think of a cochain, which is a function defined on the cells of the manifold, with values in some abelian group, called the coefficient group. Thus a cochain is very different from a chain, although when there are only finitely many simplexes in a given dimension (and we use the same coefficient group in both cases, for example the group of integers), then there is an unnatural isomorphism between the chain group and the cochain group in that dimension.

This question of 'naturality' is what ultimately justifies the modern viewpoint about homology and cohomology. First explicitly formulated by Eilenberg and Mac Lane in 1945, the functorial approach is now so well established that newcomers to topology scarcely realize that it had to be consciously chosen and by no means commanded universal assent at its inception. Now we understand that homology provides a 'map' of topology into algebra, in that spaces X are replaced by (graded) abelian groups H_*X and continuous functions $f\colon X \to Y$ are replaced by homomorphisms $H_*f\colon H_*X \to H_*Y$; and cohomology provides a 'reverse' or 'contravariant' map, in that spaces are replaced by abelian groups H^*X and continuous functions $f\colon X \to Y$ by homomorphisms $H^*f\colon H^*Y \to H^*X$. These 'maps' preserve identity transformations and respect composition in the sense that, if also $g\colon Y \to Z$, then $H_*(gf) = (H_*g)(H_*f)$, $H^*(gf) = H^*(f)H^*(g)$.

But we are anticipating and should return to our outline of the evolution of algebraic topology. In the middle 1930s came two major algebraic developments. One was the realization that the cohomology structure was richer than simply that of a graded abelian group. There was a natural ring structure which was commutative in the graded sense. It was easy to find examples of topological spaces which were not distinguished by their cohomology groups but which were distinguished by their cohomology rings. A second crucial advance at this time was the invention by Hurewicz[1] of the homotopy groups. These are generalizations of the fundamental group; the nth homotopy group, $\pi_n X$, of the space X is the set of classes of maps of the n-sphere S^n into X, endowed with a natural group structure. However, for $n \geqslant 2$, the homotopy groups are all commutative, whereas, if $n = 1$, we have the fundamental group, which need not be commutative.

Major advances in algebraic topology before the Second World War were made by Henry Whitehead. Among his contributions were his insistence on the importance of homotopy type rather than homeomorphism. Two spaces X and Y are said to be of the same homotopy type if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that the composite gf is homotopic to the identity on X and the composite fg is homotopic to the identity on Y. The homology groups, the cohomology ring, and the homotopy groups are all not only topological invariants but homotopy invariants. A second major contribution

[1]The invention should, strictly speaking, be credited to Čech; but he found no application for them, so that they lay dormant for some four years.

by Henry Whitehead was his invention of the so-called CW-complexes. These are topological spaces endowed with a combinatorial structure which, while it is much more general than the structure of a simplicial complex, is nevertheless usually much easier to handle. In particular one can usually get by with a very few cells in a CW-complex where one would have to have a large number of simplexes in a simplicial complex built on the same topological space. Thus, for example, the most natural simplicial triangulation of the torus requires 9 0-simplexes, 27 1-simplexes and 18 2-simplexes; whereas the natural CW-structure requires 1 0-cell, 2 1-cells and 1 2-cells.

A third major contribution of Henry Whitehead was a certain binary operation, now always called the Whitehead product. This is an operation on pairs of elements of the homotopy groups which is in many respects analogous to the product structure in cohomology. Indeed it can be said that the Whitehead product enables one to define a Lie ring structure in homotopy. The Whitehead product is homotopy invariant.

Finally, in this incomplete list of his contributions, let us mention the exact homotopy sequence. Not only was this a major discovery in itself but it heralded a far more sophisticated approach to the study of algebraic invariants of homotopy type. This approach was taken much further after the war, principally by Steenrod and his successors who developed a systematic theory of *cohomology operations*, that is to say, operations on the cohomology of a space which are homotopy invariant, and by Serre, who showed how to exploit spectral sequences in homotopy theory, a massive generalization of the notion of an exact sequence. To be sure there are also *homotopy operations*, but these, at any rate initially, did not seem anything like so rich. One obvious reason for this was that we had arbitrary coefficients for the cohomology groups (in particular, the rationals and the integers modulo p, for any prime p) whereas it was only somewhat later that coefficients were introduced into the homotopy groups.

The cohomology operations–for example, the Steenrod squares and the Steenrod reduced powers–and the homotopy operations–for example, the Whitehead product–are universally defined operations, in the sense that they are defined on any cohomology classes or n-tuple of cohomology classes of the required kind, or on any homotopy elements or n-tuple of homotopy elements. As algebraic topology became more sophisticated, and in response to ever deeper questions, higher order cohomology operations were studied. These are operations which are only partially defined and which take values only modulo a certain indeterminacy. A notable triumph in this regard was Adams' celebrated first proof of the nonexistence of elements of Hopf invariant one in the homotopy group $\pi_{2n-1}(S^n)$ for $n \neq 2, 4$ or 8.

This whole apparatus of algebraic topology may then be used to study obstruction problems. The pioneer here was Eilenberg who in 1940 showed how to express the obstruction to the extension of a map by means of cocycles and cohomology classes. As Steenrod pointed out, so many problems in mathematics may be presented as extension problems or lifting problems. Thus for example if X is a topological space and A is a subspace of X, and if we have a map $f: A \to Y$, it is natural to ask if we can extend f to the whole of X or, at any rate, beyond A. In a different context, given a homomorphism of groups $\varphi: G \to H$ and a surjective homomorphism $\psi: K \to H$, can φ be lifted into K?

In topology, obstruction problems generally appear as extension problems, lifting problems, cross-section problems (a specific lifting problem, where the projection of a fiber bundle is the surjection map in question), or as compression problems. The technique for studying these problems is then to employ the algebraic invariants of homotopy type such as have been described above. Since the obstruction problems fall into two dual classes it is natural that the obstructions are in certain cases formulated as being elements of cohomology groups with coefficients in a homotopy group and, in the dual cases, as elements of homotopy groups with coefficients in a cohomology group. We may represent the dual problems in their simplest form by the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & B \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle h?} & \\ & C & \end{array} \qquad\qquad \begin{array}{ccc} A & \overset{g?}{\dashrightarrow} & B \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle h} & \\ & C & \end{array}$$

Given f, g, does there exist h such that $f = hg$? Dually, given f, h, does there exist g such that $f = hg$? The extension problem takes the first form, the lifting problem and the compression problem take the second, dual form. Of course, we may refine our questions. In the first form, we may only ask that hg be homotopic to f; we may seek a classification of all possible homotopy classes h. We may relativize the problem by replacing the spaces A; B; C in our diagram by pairs A, A_0; B, B_0; C, C_0, consisting of a space and a (suitable) subspace.

The functorial approach makes it plain how we can use homology or cohomology to study these problems. In its crudest form, we may apply homology to the first question, to translate it into a question in abelian group theory. For if h exists such that hg is homotopic to f, then there must exist a homomorphism φ such that $\varphi(H_*g) = H_*f$,

$$\begin{array}{ccc} H_*A & \xrightarrow{\;H_*g\;} & H_*B \\ & {\scriptstyle H_*f}\searrow \quad \swarrow{\scriptstyle \varphi} & \\ & H_*C & \end{array}$$

namely, $\varphi = H_*h$. Thus if no such φ exists, no such h exists. This elementary conclusion is enough to show that a ball V^{n+1} may not be retracted onto its boundary S^n. For such a retraction h would give rise to a commutative diagram

$$\begin{array}{ccc} H_*S^n & \xrightarrow{\;H_*i\;} & H_*V^{n+1} \\ & {\scriptstyle \mathrm{id}}\searrow \quad \swarrow{\scriptstyle \varphi} & \\ & H_*S^n & \end{array}$$

where i: $S^n \subseteq V^{n+1}$ is the inclusion, and if we look at homology in dimension n, this becomes the (hypothetical!) commutative diagram

$$\begin{array}{ccc} \mathbf{Z} & \longrightarrow & 0 \\ & {\scriptstyle \mathrm{id}}\searrow \quad \swarrow & \\ & \mathbf{Z} & \end{array}$$

an obvious absurdity. Thus no such h exists; from this the Brouwer fix-point theorem is an immediate consequence.

Of course such crude techniques are not usually powerful enough. That is why we need an obstruction theory of considerable sophistication–hence Baues' text. In this sophisticated theory we attempt to extend through several stages (or lift through several stages), the procedure leading to higher order obstructions, expressed by means of higher order operations.

Baues has certainly given us the most systematic and comprehensive treatment of obstruction theory that has ever been attempted. Not only are the obstructions described and related to the appropriate cohomology or homotopy operations but also, where the obstructions may be overcome, Baues gives a complete classification of the classes of extensions or liftings that arise. The duality, as we have said, is explicitly and firmly exploited. Relativization is introduced to the extent that it is appropriate in order to achieve the most useful generality. There are numerous examples and illuminating applications.

Indeed some might say that Baues has told us more about obstruction theory than we really want to know. Certainly it may very well be argued that he should have exploited the duality even more and thereby somewhat shortened the text. The inevitability of the development should scarcely be regarded as a fault. Naturally in such a systematic treatment of its theme there are no real surprises–until one comes to some of the best applications. Baues leaves the reader in no doubt that this particular branch of algebraic topology has come of age.

The reader should be warned that there are numerous misprints and some disturbing solecisms; among the latter there may be those due to the fact that the author is not writing in his first language. We select, as examples of the former, three which are particularly troublesome, since they come so near the beginning of the text. There is the misprint on p. 1 where we have f: $X \to Y$ instead of f: $Y \to X$; on p. 3 where a pullback diagram is given and is meticulously described as a pushout diagram; and on p. 66 where there is an important reference to item (1.2.30) although no such item exists. Further, it is misleading to state very emphatically on p. 1 that "from now on all spaces are pointed" and to follow this with four pages on which the spaces are not pointed. A typically troublesome phrase is that on p. 51, "Analogous statements hold for the following operation." It is not clear what statements are being provided with implicit analogies, nor precisely what operation is in question. The language difficulty is doubtless responsible for the author using, on p. 4, the phrase "Cocartesian and cartesian diagrams can be combined," when he means "Cocartesian diagrams can be combined and so too can cartesian diagrams." The reader will also have difficulty in many places where letters which should be adorned with bars or tildes (thus, $\bar{H}$ or $\tilde{C}$) appear without their adornments, and thus look as if they mean something else.

This brief sample of small errors of typography and presentation are given to warn the reader that he will need to study the text very carefully, not only for its mathematical content. They are not intended to disparage the undoubted value of this book to all those concerned with this important branch of algebraic topology.

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Harmonic analysis on real reductive groups, by V. S. Varadarajan, Lecture Notes in Math., vol. 576, Springer-Verlag, Berlin, Heidelberg, New York, 1977, 521 pp., $17.20.

The study of harmonic analysis on real semisimple Lie groups has proceeded in three major currents. One of these has been via the study of more general locally compact groups. There are, for example, general results about character theory, induced representations, and the Plancherel theorem. Good references for this work are the books of Dixmier and Mackey [3], [8].

In the other extreme, there have been studies of specific groups, most frequently $SL(2, R)$, the group of 2×2 real matrices with determinant one. Although this is the simplest possible example of a semisimple real Lie group, studies of $SL(2, R)$, two of the earliest being papers by E. Wigner in 1939 and V. Bargmann in 1947, provided invaluable inspiration for the general theory [2], [13].

Finally, there has been the study of real semisimple Lie groups in general. The pioneering work in this area is due almost entirely to Harish-Chandra. This work exploits the rich structure theory of semisimple groups and the connections between analysis on these groups and the abelian Fourier analysis on their Lie algebras and Cartan subgroups.

The building blocks of harmonic analysis are irreducible unitary representations. The set $\hat{G}$ of equivalence classes of irreducible (continuous) representations of a locally compact group G is called the unitary dual of G. For $\pi \in \hat{G}$ and $f \in L^1(G)$, the operator-valued Fourier transform of f is given by $\pi(f) = \int_G f(x)\pi(x)\,dx$, where dx is Haar measure on G. The scalar-valued Fourier transform is $\hat{f}(\pi) = \operatorname{trace} \pi(f)$, if $\pi(f)$ has a well-defined trace as an operator on the representation space.

A Plancherel measure for G is a positive measure on $\hat{G}$ such that for $f \in L^1(G) \cap L^2(G)$, $\|\pi(f)\|$ is finite for μ-almost all $\pi \in \hat{G}$, and $\int_G |f(x)|^2 dx = \int_{\hat{G}} \|\pi(f)\|^2 d\mu(\pi)$. Here $\|\cdot\|$ denotes the Hilbert-Schmidt norm. Plancherel's theorem says that for a large class of locally compact groups (including compact and semisimple Lie groups, see [3]), Plancherel measure exists on $\hat{G}$, and is unique once a Haar measure dx on G is fixed. An easy