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BOOK REVIEWS

Ordered vector spaces and linear operators, by Romulus Cristescu, Abacus Press, Turnbridge Wells, Kent, England, 1976, 339 pp., $43.50. Translated from the Romanian (Editura Academiei, Bucharest, 1970) by Silviu Teleman.

An ordered vector space is just that–a set with both a (real) vector space structure and an order relation which satisfy desirable compatibility conditions. Specialization by requiring that the least upper bound (and hence the greatest lower bound) of any two elements of the space also be in the space yields a more useful object, the vector lattice (also known as a Riesz space). Most of the common spaces encountered in analysis are of this type: the (real) continuous functions on a topological space, Lebesgue spaces, Orlicz spaces, Banach function spaces, spaces of real sequences, and spaces of measures; moreover, many operator spaces also have this type of structure. As in most areas of mathematics, ordered vector spaces proceed in a two-fold manner: the development of the general theory as a subject in its own right and the use of vector lattices in analysis, primarily in the problems of Banach spaces. Since the book under review concentrates primarily on the former, we won't discuss applications here. (See, for example, H. H. Schaefer, *Banach lattices and positive operators*.) The general theory is concerned with the interplay of the order structure and the algebraic structure and with their relationship to topological structures on these spaces.

The intrinsic structure of ordered vector spaces and of vector lattices is of course a first consideration. The Riesz decomposition theorem is an example of such a fundamental theorem. We need some definitions. A *lattice ideal* in a vector lattice is a vector subspace which is also solid ($x \in M$, $y \in V$, $|y| \leqslant x \Rightarrow y \in M$). An ordered vector space is *order complete* if every upwards directed subset which is bounded above has a least upper bound. A lattice ideal in an order complete vector lattice is called a *band* if it contains the least upper bound of every subset of the ideal which is bounded above in the space. Finally two elements x, y of a vector lattice are *orthogonal* (or disjoint) if the greatest lower bound of their absolute values is zero. Call $A^{\perp}$ the set of elements y in the vector lattice such that y is orthogonal to x for each $x \in A$. The Riesz theorem says that for any subset A of an order complete vector lattice V, V may be decomposed into the order direct sum of the bands $A^{\perp}$ and $A^{\perp\perp}$; i.e. given any $v \in V$ there is a unique v_1 in $A^{\perp}$ and v_2 in $A^{\perp\perp}$ such that $v = v_1 + v_2$, and $x \geqslant 0$ implies that $x_1 \geqslant 0$ and $x_2 \geqslant 0$. Moreover, $A^{\perp\perp}$ is the smallest band containing A. (Thus the projections onto the bands are not only linear mappings, they are positive.)

Another structure theorem in this vein is the important spectral theorem due to Freudenthal (and independently to Nakano). If a vector lattice has a decomposition as above, $E = Q \oplus Q^{\perp}$, then the *projector* $[Q]$ is defined to be the projection map. The *principal projector* $[v]$ associated with $v \in E$ is the projector $[v^{\perp\perp}]$. If a vector lattice is σ-complete (order complete for countable subsets) and has a unit (a positive element u such that $u \perp x = 0 \Rightarrow x = 0$) then a form of the spectral theorem represents every member of the lattice as an integral, viz.

$$x = \int_{-\infty}^{\infty} \lambda dg_x(\lambda)$$

where the spectral function of x is defined via projectors as

$$g_x(\lambda) = [(\lambda u - x)_+]u.$$

Not only is this an important structure theorem for vector lattices but there are applications to diverse situations in analysis, including of course the expected one in operator theory.

Another important area of consideration is that of convergences and topologies defined naturally in terms of the order. First of all, a net $(x_\alpha)_{\alpha \in A}$ in an ordered vector space V is said to *order converge* to x if there is a family $(u_\alpha)_{\alpha \in A}$ in V which is directed downwards, has greatest lower bound 0, and has $|x_\alpha - x| \leqslant u_\alpha$ for all $\alpha \in A$. In terms of this convergence one is led to the notion of order-continuity. The natural question to ask: is this convergence the convergence with respect to a topology? The answer, in general, must be no since order convergence does not have the property that $x_n \rightarrow x$ in order iff every subsequence of (x_n) has a subsequence converging to x. (For an example, consider the space of measurable functions on a finite measure space where order convergence is given by almost everywhere convergence plus domination of the sequences by a single function.) There is nonetheless a natural order topology: the strongest locally convex topology for which every order-bounded set is bounded in this topology. A major result here is that if V is a complete ordered topological vector lattice whose topology is locally convex and metrizable, then the given topology of V is the order topology.

Questions of duality are important and somewhat complicated in our present context. First of all, the space of all order-bounded linear functionals on an ordered vector space may be larger than the span of the positive linear functionals. (Said span is usually called the *order dual*.) Secondly, the space of order-continuous linear functionals may not be the same as either of the above. Finally, in the case of an ordered topological vector space, the continuous linear functionals may be different. Relations between these dualities and conditions for coincidence can be established. For example, if V is a vector lattice, the order dual coincides with the space of order-bounded linear functionals and moreover this dual is order complete. In this case the order continuous linear functionals form a band in the order dual which is the range of some positive linear projection. In the case of normed lattices, the topological dual is contained in the order dual and is order complete. In fact, the two duals coincide if the normed lattice is additionally a Banach space. The problems of dealing with operators are of course more substantial but will not be discussed here.

What about the volume under review? Since F. Riesz announced in 1928 his work on the order dual of an ordered vector space, there has been considerable activity in the area (primarily in the USA, the USSR, and Japan), yet not many textbooks have appeared on the subject. To be sure, some functional analysis books have included material, for example M. M. Day's *Normed linear spaces* (1957) treats in a compressed fashion ordered vector spaces, vector lattices, and normed lattices and even gives Kakutani's results on abstract L- and M-spaces (vector lattices which look like L^1 and $C(\Omega)$). K. Yosida's *Functional analysis* (1965) has a chapter on representations of vector lattices. But, in general, questions of order have been presented as subsidiary to their uses (often implicit) in the study of Lebesgue spaces, Orlicz spaces, Banach function spaces, and continuous function spaces. A few exceptions occurred outside the United States: H. Nakano's *Modular semi-ordered linear spaces* (1950) and *Functional analysis in partially ordered linear spaces* (1950) by L. V. Kantorovitch, B. Z. Vulikh, and A. C. Pinskh were published in Japan and the Soviet Union, respectively. In Romania the forerunner of the book under review was published in 1959. This was revised and expanded for republication in 1970, and then updated and translated into English for the current volume.

The author presupposes "some general facts" from functional analysis: his 1970 text, *Functional analysis* (Romanian) is recommended, whereas "... for the English-speaking reader, the treatise *Linear operators*, by N. Dunford and J. Schwartz (Vol. I), is an excellent reference." The level of preparation required is of course nowhere near as encyclopedic as that provided by Dunford and Schwartz, and in fact most of what is needed is touched upon in its first two chapters.

Chapter 1 covers the standard material on ordered sets concisely. Included is Stone's representation of Boolean lattices. The notions of order convergence of sequences and generalized sequences are here also. Chapter 2 is a quick review of the necessary facts from topological vector spaces.

The main material really starts in Chapter 3 where ordered vector spaces and vector lattices are introduced. A nice discussion of orthogonal complements, band decompositions, and projectors is included. (Caution: the words "cone", "band", and "lattice ideal" do not appear in this book, although of course the concepts do.) In addition, a small collection of examples appears in a separate section. Chapter 4 deals with vector lattices with special properties: lattices with units of various kinds, σ-complete lattices, complete lattices. The famed Freudenthal spectral theorem appears as just another theorem with no indications given of its importance or uses.

The study of operators begins in Chapter 5–specifically, it deals with regular operators (an operator is *regular* if it is the difference of two positive operators) and order-continuous operators. This of course is the order dual idea generalized to operators. Some material is presented on linear functionals and bilinear operators as well.

Chapter 6 introduces topological ordered vector spaces, i.e. an ordered vector space with a topology that is compatible with both the order and the vector space structure (which are of course already compatible). Theorems are given so that one is able to recognize this compatibility when it occurs. The various kinds of spaces of linear functionals are compared–the order-bounded, the regular, the order-continuous, and the topology-continuous. A short discussion of normed vector lattices leads to the L^p spaces and the abstract L- and M-spaces. Kakutani's concrete representation of L-spaces is given, but strangely enough the corresponding one for M-spaces does not appear.

A novel topic appears in Chapter 7–linear spaces with norm-functions that are vector-lattice valued. The results are reminiscent of those in vector-valued function spaces. In particular, integral representations are obtained for bounded linear operators on the space of continuous vector-valued functions and on the space of (Bochner) integrable vector-valued functions (where the domain of these functions is a compact interval on the real line). The Hellinger integral is developed for this last purpose. Finally, integration of vector-valued functions with respect to vector measures (via a bilinear map) is presented.

The last chapter "... gives a brief exposition of the manner in which the theory of ordered vector spaces can be used in various branches of mathematics." These include operator equations in various contexts, operator extensions, the spectral theorem for selfadjoint operators on Hilbert space, and fixed points for positive contractions.

The book is well organized and clearly written. The level of exposition is detailed, yet important material is easily accessible (if one already knows what's important–there are no real indications of the high points). Each chapter ends with bibliographic notes which reference and complement the text material. The biggest drawback of the book is that there are no exercises or problems whatever, and very few examples. In fact, the whole circle of motivating examples available in Orlicz spaces, Banach function spaces, normed Köthe spaces is not mentioned at all. In addition, with the exception of work by the author, the bibliography stops at 1970 and omits the three major recent books on the subject, viz., G. Jameson's *Ordered linear spaces* (1970), A. L. Peressini's *Ordered topological vector spaces* (1967) and the first volume of the very substantial and important *Riesz spaces* (1971) by W. A. J. Luxemburg and A. C. Zaanen.

Nonetheless, the book includes a good selection of material organized in a usable fashion and would make a good reference and text, if properly supplemented.

NEIL E. GRETSKY

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Subharmonic functions, by W. K. Hayman, and the late P. B. Kennedy, Vol. I, Academic Press, London, New York, San Francisco, 1976, xvii + 284 pp., \$25.50.

Subharmonic functions have been around for a long time, although not known by that name originally, and have played a central role in the development of mathematics. The Newton and Coulomb inverse square laws for gravitational and electromagnetic forces, respectively, made this role