BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 1, January 1979

RESEARCH ANNOUNCEMENTS

CLASSIFICATION OF THE IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{sl}(2, \mathbf{C})$

BY RICHARD E. BLOCK

Let $\mathfrak{g}$ be a nonabelian Lie algebra over an algebraically closed field K of characteristic 0. One is interested in the (algebraically) irreducible representations of $\mathfrak{g}$ acting on a vector space which is allowed to be infinite dimensional. The subject of enveloping algebras is largely concerned with these, but even in the simplest nonabelian case, with $\mathfrak{g} = \mathfrak{h}$ the 3-dimensional (nilpotent) Heisenberg algebra, as Dixmier remarks in discussing the situation when $K = \mathbf{C}$ in the preface to [2], "a deeper study reveals the existence of an enormous number of irreducible representations of $\mathfrak{h}$ . . . It seems that these representations defy classification. A similar phenomenon exists for $\mathfrak{g} = \mathfrak{sl}(2)$, and most certainly for all noncommutative Lie algebras."

However, as we shall see, the situation for $\mathfrak{h}$ and for $\mathfrak{sl}(2)$ turns out to be far nicer than hoped for. Indeed we announce here a determination and classification of all irreducible representations of $\mathfrak{h}$, of $\mathfrak{sl}(2)$, and of the 2-dimensional nonabelian Lie algebra, and thus of the prototypes respectively of nilpotent, simple, and solvable Lie algebras. As a guide to the meaning of "classification" and because our results use the same invariants, consider a classical situation of an (associative) algebra for which the irreducible representations have long been classified, namely, the algebra B of formal linear differential operators with rational function coefficients, i.e., $B = K(q)[p]$, the (noncommutative) polynomials in an indeterminate p where multiplication is determined by the relation $pq - qp = 1$. Then B is a left principal ideal domain. Therefore [3] a B-module M is simple if and only if $M \cong B/Bb$ for some $b \in B$ which is irreducible (i.e., $b = ac$ implies a or c is a unit); and $B/Bb \cong B/Ba$ if and only if a and b are similar, i.e., there exists $c \in B$ such that $(b, c) = 1$ and $a = [b, c]c^{-1}$ where (b, c) is a

Received by the editors August 14, 1978.
AMS (MOS) subject classifications (1970). Primary 17B10, 17B35, 16A64; Secondary 16A04, 16A08, 16A18.
[1]Research supported in part by grant NSF MCS 76-11413.

0002-9904/79/0000-0020/$02.00

right g.c.d. and $[b, c]$ a left l.c.m. (which always exist) (similar is the noncommutative generalization of associate).

The subalgebra $K[q][p]$ of B generated by p, q is the Weyl algebra A_1. Since $A_1 \cong U\mathfrak{h}/U\mathfrak{h}(z-\alpha)$ for $0 \neq z \in$ center $\mathfrak{h}$ and $0 \neq \alpha \in K$, the problems of finding the irreducible representations for A_1 and for $\mathfrak{h}$ are equivalent. Our solution for this problem as well as for $\mathfrak{sl}(2)$ involves the new notion of *preserving,* defined in terms of certain polynomials which we now introduce. For $\alpha \in K$ let μ_α denote the valuation of $K(q)$ determined by the prime $q-\alpha$ of $K[q]$, and extend μ_α to a function (also denoted μ_α) on B by setting $\mu_\alpha(\Sigma_j b_j(q)p^j) = \min\{\mu_\alpha(b_j(q)) - j | j \geqslant 0\}$. Then define $\theta_{\alpha,b}(\lambda) \in K[\lambda]$ $(\alpha \in K$, $b = \Sigma_j b_j(q)p^j \in B)$ by

$$\theta_{\alpha,b}(\lambda) = \sum_j \{((q-\alpha)^{-\mu_\alpha b - j} b_j(q))(\alpha)\}(-1)^j \lambda(\lambda+1)\cdots(\lambda+j-1).$$

(It can be proved that μ_α is a valuation on B, and extends to a valuation on the quotient division ring whose residue field is $K(\lambda)$; then with φ_α the corresponding place, $\theta_{\alpha,b}(\lambda) = \varphi_\alpha((q-\alpha)^{-\mu_\alpha b} b)$.) Call b *α-preserving* if $\theta_{\alpha,b}(\lambda)$ has no positive integral root, and *preserving* it is α-preserving for all $\alpha \in K$. It can be shown that b is preserving if it is α-preserving for a certain *finite* set of α's, in particular (when b is normalized to be in A_1) for the set of roots of the leading coefficient $b_r(q)$; thus if $K = \mathbf{C}$ the property of b being preserving is computable given the roots of $b_r(q)$.

The A_1-module $(K[p], q-\alpha$ acts as $-d/dp)$ is simple and is precisely the simple A_1-module for which q has α as an eigenvalue.

THEOREM 1. *If $a \in B$ is irreducible and preserving then the A_1-module $A_1/A_1 \cap Ba$ is simple.*

THEOREM 2. *If M is a simple A_1-module then either $M \cong (K[p], q-\alpha$ acts as $-d/dp)$ for some $\alpha \in K$ or $M \cong A_1/A_1 \cap Ba$ for some a as in Theorem* 1.

Since the $A_1/A_1 \cap Ba$ above have no eigenvector for q, the following completes the classification of the simple A_1-modules.

THEOREM 3. *Two simple A_1-modules $A_1/A_1 \cap Ba$, $A_1/A_1 \cap Bb$ are isomorphic if and only if a and b are similar* (*in B*).

Now consider the case of $\mathfrak{g} = \mathfrak{sl}(2, K) = \mathfrak{S}$, with canonical basis e, f, h. For $\beta \in K$ the map $e \to q$, $h \to 2qp - \beta$, $f \to -(qp-\beta)p$ extends to a homomorphism ρ_β of $U\mathfrak{S}$ to B. The simple $\mathfrak{S}$-modules for which e has an eigenvector v (with eigenvalue α) are as follows: if $\alpha = 0$, the highest weight modules $L(\beta)$ $(\beta \in K)$ (with $hv = \beta v$); if $\alpha \neq 0$, the simple Whittaker module $\mathrm{Wh}_\beta(\alpha)$ (see [1], [4]; $\alpha = \eta(e)$), with basis $t^0 = v, t^1, \ldots$ where $ht^i = 2t^{i+1}$, $et^i = \alpha(t-1)^i$, $ft^i = \alpha^{-1}(t+1)^i(-t-t^2+(\beta^2+2\beta)/4)$. The only isomorphisms among these are $\mathrm{Wh}_\beta(\alpha) \cong \mathrm{Wh}_\delta(\alpha)$ whenever $\beta^2 + 2\beta = \delta^2 + 2\delta$. For any $\beta \in K$ write β' for the other root of $\lambda^2 + 2\lambda = \beta^2 + 2\beta$ i.e., $\beta' = -\beta - 2$.

THEOREM 4. *Suppose $a \in U\mathfrak{s}$, $\beta \in K$, $\rho_\beta a$ is irreducible (in B) and $\rho_\beta a$ and $\rho_{\beta'} a$ are preserving. Then the $U\mathfrak{s}$-module $\rho_\beta U\mathfrak{s} / \rho_\beta U\mathfrak{s} \cap B(\rho_\beta a)$ is simple.*

THEOREM 5. *If M is a simple $U\mathfrak{s}$-module then either $M \cong L(\beta)$ for some $\beta \in K$ or $M \cong \mathrm{Wh}_\beta(\alpha)$ for some $\alpha, \beta \in K$, $\alpha \neq 0$, or $M \cong \rho_\beta U\mathfrak{s} / \rho_\beta U\mathfrak{s} \cap B(\rho_\beta a)$ for some a as in Theorem 4.*

Again the following completes the classification.

THEOREM 6. *Two simple $U\mathfrak{s}$-modules $\rho_\beta U\mathfrak{s}/\rho_\beta U\mathfrak{s} \cap B(\rho_\beta a)$, $\rho_\delta U\mathfrak{s}/\rho_\delta U\mathfrak{s} \cap B(\rho_\delta b)$ are isomorphic if and only if $\beta^2 + 2\beta = \delta^2 + 2\delta$ and $\rho_\beta a$ and $\rho_\beta b$ are similar (in B).*

Analogous results hold for the 2-dimensional nonabelian Lie algebra, realized say as the subalgebra $\mathfrak{b} = Kh + Ke$ of $\mathfrak{s}$, with the following changes: the simple $\mathfrak{b}$-modules for which e has an eigenvector are $\mathrm{Wh}_0(\alpha)$ (for $\alpha \neq 0$) and, for each $\delta \in K$, $Kv \subseteq L(\delta)$; restrict β to 0 and change the condition on preserving to the condition that $\rho_0 a$ be preserving and $\theta_{0,\rho_0 a}(\lambda) \in K$ (or equivalently, $a = e^u(ec + \alpha)$ for some $u \in \mathbf{N}$, $c \in U\mathfrak{b}$ and $0 \neq \alpha \in K$).

The ring B is the localization of its subrings A_1 and $\rho_\beta U\mathfrak{s}$ with respect to the multiplicative subset $S = K[q] - \{0\}$.

THEOREM 7. *Every simple B-module N contains a unique simple A_1-submodule ψN and, for every $\beta \in K$, a unique simple $\rho_\beta U\mathfrak{s}$ submodule $\psi_\beta N$; ψN (resp. $\psi_\beta N$) is contained in every nonzero A_1- (resp. $\rho_\beta U\mathfrak{s}$-) submodule of N. Also ${}_B N \cong S^{-1}(\psi N) \cong S^{-1}(\psi_\beta N)$, and if M is a simple S-torsionfree A_1-(resp. $\rho_\beta U\mathfrak{s}$-) module then $\psi(S^{-1}M)$ (resp. $\psi_\beta(S^{-1}M)) \cong M$. Thus the map $N \longrightarrow \psi N$ (resp. $N \longrightarrow \psi_\beta N$) sets up a bijection between the set of isomorphism classes of simple B-modules and the set of isomorphism classes of S-torsionfree simple A_1-modules (resp. $U\mathfrak{s}$-modules with the Casimir element $4fe + h^2 + 2h$ acting as $\beta^2 + 2\beta$).*

Here is a formula involving the $\theta_{\alpha,b}(\lambda)$ which helps to explain their relevance to modules. If $a \in A$, then for the action on ($K[p]$, $q - \alpha$ acts as $-d/dp$), for every positive integer s we have

(1) $$(q-\alpha)^{-\mu_\alpha a} a \cdot p^{s-1} = \theta_{\alpha,b}(s)p^{s-1} + \text{lower terms.}$$

Somewhat similar formulas hold for the actions of $U\mathfrak{s}$ on $\mathrm{Wh}_\gamma(\alpha)$ and $L(\delta)$. The proof of Theorem 1 begins by showing that a maximal ideal J of A properly containing $A \cap Ba$ intersects S, and so q has an eigenvector on A/J. Then one uses α-preserving and (1). Theorem 4 is similar. The remaining theorems use properties of minimal annihilators and localizations. Theorems 2, 5 and 7 also depend on the following.

LEMMA. *If $b \in B$, there exists $d \in S$ such that* bd^{-1} *is preserving.*

The proof of Theorem 7 also uses Theorem 1 and 4; if $N = B/Bb$ where b is preserving then $\psi N = (A_1 + Bb)/Bb$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CALIFORNIA 92521