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Topological algebras, by E. Beckenstein, L. Narici, and C. Suffel, North-Holland, Amsterdam, New York, Oxford, 1977, xii + 370 pp., $24.50.

An algebra, in the present context, is an associative linear algebra over a field K, and a topological algebra is such an algebra with a suitably related topological or quasi-topological structure. This statement requires, of course, considerable clarification and qualification; some of this (but by no means enough) will appear in the course of the review. Very often the field K is R or C, and in the book of Beckenstein, Narici and Suffel this restriction is observed throughout the text; some generalisations are envisaged in the exercises. There are indeed problems in abundance in the classical cases, and there is much to be said for restricting attention to these cases in a study such as this.

Examples of topological algebras are widespread in analysis. The primary example is, of course, provided by the normed algebras (including various special types, often separately designated, such as function algebras, operator algebras, and the like). These, at least when complete (i.e., Banach algebras), have long been established as a useful tool in analysis, as well as rewarding subjects for study in their own right. And here we encounter a slight terminological ambiguity; sometimes the term topological algebra is used to include explicitly the class of normed algebras (there are proceedings of symposia on topological algebras that have included little else), and sometimes to include them only implicitly, and with subordinate status. This latter convention is probably the more convenient, and is followed in the book and its review. The situation is not unlike that of the real numbers as a subset of the complexes: it is often convenient to take the reals as known, and then develop the theory of the complexes. Here it is assumed that standard results in normed algebras are known, and available: the main preoccupation is building on that foundation an appropriately generalised structure. It is not in fact true that everything in topological algebra theory is a generalisation of the normed case; not all topological vector spaces are generalisations of normed spaces. But little will be lost by focussing attention on topological algebras as generalisations of normed algebras.

Banach algebras were developed in the late 1930s (mainly by Gelfand and his associates, although there were others also). In a very few years they had become an indispensable part of modern analysis. To mention only one application, existence theorems in harmonic analysis are very conveniently based on results from Banach algebra theory. Moreover, the intrinsic simplicity of the Banach algebra machinery enables it to be used effectively to clear away irrelevant difficulties in harmonic analysis problems, leaving the core of 'hard' (often, not all that hard!) analysis to be tackled by other means. About the same time, the theory of topological vector spaces was being worked out. Banach spaces, within a few years of their formal introduction, had turned out to be rather restrictive for the needs of analysis, and were duly generalised. This theory of topological vector spaces (locally convex spaces, in particular) was appropriate for many applications. One need look no farther than the theory of generalised functions (= distributions) for a major example. Already, it should perhaps be noted, it was clear no single generalisation was appropriate for all contexts. Indeed, one of the main themes of the whole development was the realisation that the machinery should usually be set up in conformity with the problem. The Hilbert space approach, where one seeks to put all problems into a standard framework, gave way to a more pluralistic (anarchistic?) way of life. In any event, with all these developments in progress, it was very natural to suppose that some fusion of the two lines, in a theory of topological algebras, would produce something interesting and important. And so it has, up to a point; but to a much more limited extent, and with much less impact, than seemed likely at the time (a quarter of a century and more ago) of the early developments. It may perhaps be worth while to pause for a moment to try to identify the reasons. The basic one is probably that the major motivation for studies in topological algebras has been, and remains, internal rather than external. There are few problems currently studied where a theory of topological algebras more general than Banach algebras would be of clear advantage and immediate applicability. That is not to say that such problems do not exist (holomorphic functions of restricted growth form one important area); but they are not numerous. Another reason may be that the primary examples of commutative topological algebras that are not Banach algebras are already presented as algebras of functions, so that a representation theory is scarcely an immediate necessity. And another may well be the virtual absence in general (nonmetrisable) topological algebras of 'automatic continuity' results (to which we return at the end of the review), such as are available in Banach algebras. This results in a much less clean and appealing development in some respects: supplementary hypotheses seem to be rather frequently required.

To effect a marriage of linear algebras with topological vector spaces requires two decisions. First, how general should be the vector space topology? and second, how should multiplication relate to the topology? As far as the vector space topology is concerned the standard choices are available, and have all been used on occasions. Naturally local convexity and/or metrisability (usually with completeness thrown in) have been most widely assumed. Even here, however, there are awkward problems; we refer to one at the end of the review. We shall here use the term Fréchet space to mean a complete metrisable topological vector space, without any assumption of local convexity: conventions unfortunately differ on this as on many other matters of terminology. The Fréchet space properties imply that even if multiplication is assumed only to be separately continuous, it must in fact be jointly continuous. More significantly, perhaps, the closed graph theorem and its consequences hold. Now, in vector space theory it has long been realised that results of this kind are true in situations more general than the classical one of Fréchet spaces (at least in the convex case); barrelled spaces and fully complete spaces are relevant here. Such generalisations in topological algebras have been considered and would presumably repay further study. For multiplication, it is natural to require from an aesthetic point of view (and with the model of a topological group in mind) that in a topological algebra the operation is jointly continuous: if x is near x_0 and y is near y_0 then xy is near x_0y_0. Normed algebras certainly enjoy this property. However, it should be remarked that this assumption of joint continuity of multiplication effectively cuts out a class of algebras that one would certainly wish to include in any comprehensive theory. Let E be a topological vector space, and $L(E)$ the algebra of continuous linear maps of E into itself. The algebra $L(E)$ may be topologised in several ways. From many points of view the most natural type of topology is where the basic neighbourhoods of 0 in $L(E)$ are

$$\{T: Tx \in N, \forall x \in B\}$$

where N is a neighbourhood of 0 in E, and B is a bounded subset of E, perhaps further restricted to belong to a subclass $\mathfrak{B}_1$ of the bounded sets $\mathfrak{B}$; for example, $\mathfrak{B}_1$ might be the compact sets in E. Now, in no such topology can multiplication in $L(E)$ be jointly continuous, apart from the case in which it is obviously so, namely when E is already a normed space (and the topology of $L(E)$ is then defined as usual). The proof is easy, and the result well known. The failure of joint continuity in operator algebras in the weak topology is, of course, familiar. So it would seem necessary to try to set up a theory of topological algebras with separate rather than joint continuity of multiplication. In fact not much has been achieved at this level of generality, with neither commutativity nor joint continuity. In the book under review both commutativity and joint continuity are assumed throughout. And in much of the work that has been done conditions more special than joint continuity (in particular m-convexity, described below) have proved useful. One device, used by J. L. Taylor in [5], deserves mention: it enables a decision about separate or joint continuity of multiplication (and some other properties) to be substantially deferred. Start with a (locally convex) topological vector space, and let $\tilde{\otimes}$ be a (general) tensor product. Now define an algebra relative to $\tilde{\otimes}$ to be a locally convex space A with an associative operation of multiplication (and a multiplicative identity) such that the map $(a, b) \to ab$ of $A \times A$ to A extends to a continuous linear map from $A \tilde{\otimes} A$ to A. By appropriate specialisations of $\tilde{\otimes}$ (e.g. to the completed inductive or projective tensor product) the corresponding algebras and modules appear with exactly the properties required.

What can reasonably be expected from a theory of topological algebras? In the first place, some kind of structure theory–perhaps not of a very detailed kind, in view of the corresponding results (or their absence) for normed algebras. One should have a notion of 'simple' topological algebra, and try to show more general algebras isomorphic to collections of simple algebras, appropriately strung together. In the commutative case the scalar field K might seem the obvious candidate for the basic building-block (some qualifications have to be made: see below)–but what about the noncommutative case? In fact, even for Banach algebras, in the noncommutative situation it is only in the presence of some extra structure (an involution) that much can be said. More should not be expected here. Although the commutative case has received its due share of attention, not much has been done in the noncommutative case until recently. Next, one might look for some systematic working through of the properties of Banach algebras with a view to characterising the classes of topological algebras that enjoy the properties in question. The kind of thing that can be done is typified by the definition of a Q-algebra. These are topological algebras in which the invertible elements form an open set (we assume an identity element: in its absence the customary arrangements or reformulations can be made). Various properties of Q-algebras are then worked out; for example, spectra are compact and maximal ideals are closed. At the end of it all one should have a good classification of topological algebras by their Banach-algebra-like properties. Another feature of a 'good' theory is the number and significance of the contacts it makes with other branches of mathematics (or–even better–with other human activities of a basically nonmathematical character!) It is probably true that topological algebras have lost something, as compared with Banach algebras, in the richness of their contacts with analysis. They may well have compensated by developing closer links with set theory and topology. Indeed a significant part of the discussion in the present book is devoted to questions of this character. It is an over-simplification–but perhaps a not entirely misleading one–to say that the book has much more of the character of Gillman and Jerison than of Gelfand, Raikov and Shilov or of Naimark. Applications to analysis of the Banach algebra type are not touched on.

The earliest work on topological algebras of the kind now considered appears to be due to Richard Arens and Irving Kaplansky in the late 1940s. This was developed further both by Arens and by E. A. Michael in his AMS memoir [4]. Here the basic condition imposed is that of multiplicative convexity (m-convexity): this has continued to be widely used in subsequent studies. It is assumed that the topology of the algebra A is given by a set of seminorms p_i (so that, as a vector space, A is in any case locally convex) satisfying the additional condition

$$p_i(xy) \leqslant p_i(x)p_i(y)$$

for all $x, y \in A$ and all indices i. This implies immediately that in A multiplication is jointly continuous. Such an algebra A is said to be locally m-convex. Now, the quotient of A by the kernel of p_i is in a natural way a normed algebra A_i, with completion B_i. If A is complete, it is exactly the projective limit of the Banach algebras B_i; in any event A is dense in this projective limit. Thus, especially in the commutative case, questions about spectra, inverses and the like can easily be reduced to the corresponding questions in the algebras B_i, where more machinery is available to facilitate an answer. It should be remarked however that the relation between A and the algebras B_i may involve unexpected complications. The obvious (and rather well-behaved) algebras of functions may be somewhat misleading examples here. There is an example, due to S. Rolewicz, (and quoted in [7]), of a commutative locally m-convex Fréchet algebra A that is semisimple, and such that for no possible choice of admissible seminorms p_i are all the algebras B_i semisimple. Michael's study did much to give direction to subsequent work on topological algebras; several of the questions he raised have become central problems, and one or two still remain unanswered.

The difficulties encountered in setting up a satisfactory representation theory of topological algebras, even in the commutative case, are already present to some extent on the purely algebraic level. Take as an example an algebra A that contains a subalgebra isomorphic to the field F of rational functions–for instance, A could be the algebra of equivalence-classes of almost-everywhere-finite measurable functions on [0, 1], with almost-everywhere pointwise operations or, indeed, A could be F itself. Suppose we try to represent such an algebra A as an algebra of (everywhere finite) functions on some–presumably large–structure space S. We must necessarily fail. Any algebra of functions on S has many characters (by a character we mean a nonzero homomorphism $A \to K$; if A is topological a character is not necessarily continuous in general): for each $s \in S$ the evaluation map $x \to x(s)$ is a character. Our algebra A has no characters at all: for if P is the subalgebra of polynomials then any character χ restricted to P is evaluation at some $t \in K$: $\chi(x) = x(t)$ for $x \in P$. If any such χ were to exist, let t be the corresponding point of K and let x be an element of P with $x(t) = 0$. Certainly x is invertible in F: and then $1 = \chi(1) = \chi(x)\chi(x^{-1}) = x(t)\chi(x^{-1}) = 0$. The conclusion is that if we are to have an adequate representation of a commutative algebra over K by X-valued functions on S, then X has to be something other than K itself, in general.

However, given that there are enough characters, and more particularly enough continuous characters, to separate the points of A, a functional representation of the familiar kind can be achieved. Such is the case, for example, for locally m-convex algebras. In any case algebras of continuous functions are of clear interest and importance as (at least) primary examples, and have received considerable attention. The first three chapters of Beckenstein, Narici and Suffel (half the book) are devoted to these algebras and some consequential problems arising from their study. The first chapter is on algebras of continuous functions in general, on a topological space, with no topology assumed on the functions. In the second chapter the functions are given the compact-open topology and studied as a vector space. The information obtained here is quite full and detailed in several directions. Topological algebras proper are introduced in the fourth chapter (and a reader could if he wished begin the book at that point). As already noted, these are always commutative and have jointly continuous multiplication. The theory is developed as far as some Gelfand-type results on functional representations, and gives a very readable account of the basics of the subject in the degree of generality selected.

The last chapter of Topological Algebras introduces a type of algebra that is not, strictly speaking, topological at all, but that certainly merits attention. In a normed vector space (or algebra) the norm balls are simultaneously neighbourhoods and bounded sets. In more general topological vector spaces the two must be distinguished; and neighbourhoods have generally been taken as primary and bounded sets then defined in terms of them. In fact the bounded sets can equally well be taken as the primary structure: there are some obvious properties that are normally taken as axioms, such as that a subset of a bounded set is bounded and the union of two (or any finite number of) bounded sets is bounded, in addition to the properties more specific to the algebra structure, that scalar multiples and (finite) sums and products of bounded sets should be bounded. Such a structure, based on bounded sets, is a bornology rather than a topology. Even in the absence of any algebraic structure at all, one can define bornologies; but it appears that only when a suitably rich algebraic structure exists, compatible with the bornology, is the concept a fruitful one. For vector spaces there is some gain in introducing a bornology, as a partial dual of a topological structure; in linear algebras the advantages are more substantial. In many contexts, boundedness of maps is a natural requirement rather than continuity, and a bornology gives the appropriate setting. In topological algebra theory, bornologies seem to have been first used systematically by Waelbroeck [6]; much subsequent work in the same direction has been done. For a recent account see [3].

It is of course very plausible that any gadget that works in Banach algebras should work also–perhaps somewhat modified–in more general topological algebras. One development in Banach algebra theory, that has been useful for some problems in measure algebras on groups, involves homology and cohomology: the basic algebraic framework can be matched with the norm structure to produce a satisfactory blend here. For some purposes–applications to functions of several (operator) variables and the corresponding spectral theory–further extensions are called for. It turns out that such are possible; the technicalities become rather formidable but the basic strategy is successful. An account of all this can be found in Taylor's paper [5] already mentioned. Another such topic is multiplier theory.

An important additional structure that an algebra may have, much used and deservedly popular in the normed case, is an involution. This is a map $x \to x^*$ of the algebra on to itself such that $x^{**} = x$, $(x + y)^* = x^* + y^*$, $(\alpha x)^* = \alpha x^*$ for $\alpha \in R$, $(\alpha x)^* = \bar{\alpha} x^*$ for $\alpha \in C$, $(xy)^* = y^* x^*$, and having such continuity properties as may be appropriate. We shall use the term star-algebra (*-algebra) for any algebra with an involution, with no assumption about continuity (Beckenstein, Narici and Suffel use the term in a different sense, as the topological analogue of a B^*-algebra). In the normed case even rather mild assumptions about the involution enable the theory to be carried a long way forward. In topological algebras that are not normed, involutions have been studied, but perhaps not as intensively as might have been expected. If A is a complete locally m-convex algebra with an involution and if the involution and the defining seminorms p are related by

$$p(x^*x) = (p(x))^2,$$

then there are good analogues of the results for B^*-algebras. In general, results on Hermitian elements and functionals extend, perhaps with substantially new proofs. In the commutative case A is isomorphic to the algebra of continuous functions on the structure space (space of continuous characters, suitably topologised), with the topology of uniform convergence on equicontinuous subsets.

Another result that can be extended from normed *-algebras to topological *-algebras (not commutative, in general) involves positive functionals. A functional f is positive if

$$f(x^*x) \geqslant 0, \qquad x \in A.$$

Then if A is a Fréchet algebra (not assumed convex), with an identity, and with a continuous involution, every positive linear functional f on A is necessarily continuous.

This last result is typical of a general class of problems–automatic continuity problems–that are as old as the subject and have been much studied in recent years, both for normed and for more general topological algebras. When is every homomorphism from one given algebra to another necessarily continuous? For a comprehensive account of recent work see [**1**]. (I am much indebted to Garth Dales for a pre-publication copy of his survey.) The seminal theorem is Gelfand's result that every character on a Banach algebra is continuous, and in fact has norm 1. The (very easy) proof of this does not, it should be noted, involve the Axiom of Choice or an equivalent axiom in any form: the axiom does come into play if we wish to assert that characters exist, or exist in abundance. In any event, it is natural then to look for other situations where every map of some algebraically defined class has good topological properties. An obvious question is the following:

> Is every character of a commutative locally m-convex Fréchet algebra continuous? (*)

This was already raised in [**4**] and no complete answer is as yet available. Recent work on other aspects of automatic continuity has directed attention to some facts that had not formerly been given much prominence, but are now in the forefront or current developments: perhaps they have relevance for (*) also. In the past, functional analysts have for the most part been talking prose (= ZFC) without being very explicit about it. Now however they have become aware of a spectrum of other linguistic possibilities, and many are seeking to relate their problems to other (perhaps unfamiliar) extensions of ZF. The attention given to Solovay's Axiom LM a few years ago has made for a heightened interest in this kind of activity. In particular, work on automatic continuity has been closely linked with the axioms of set theory being used. So instead of (*) we might rather have the reformulation

> Characterise the extension of ZF in which every character on a commutative locally m-convex Fréchet algebra is continuous.

Of course, even partial information would be welcome here, as elsewhere. (It should be remarked that if LM is assumed, then (*), and more, is true: see for example [**2**].) But it looks very much as though in the future analysts generally are going to have to think harder about the framework(s) in which they try to solve their problems. Is ZF + ?AC + ?GCH really appropriate, or might NBG or a variant be better? Before long, who knows, we may all have to acquire at least a passing acquaintance with such currently exotic concepts as κ-like trees, MA, Σ_m^n sets, and $\diamond$.

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Finite element Galerkin methods for differential equations, by Graeme Fairweather, Marcel Dekker, Lecture Notes in Pure and Applied Mathematics, vol. 34, New York, Basel, 1978, ix + 263 pp., $19.75.

Elliptic boundary value problems come up often in applications. These problems are usually solved numerically. A simple approach is to replace each derivative with a difference quotient–this is called the method of finite differences. Thus the solution of the differential equation is approximated by approximating the differential equation. Another approach is to approximate the solution of the differential equation directly. The Finite Element Method does this as follows: Consider the differential equation to correspond with a variational principle. This variational principle is solved approximately. The approximation is that a finite family of functions replaces the infinite set of functions that satisfy the boundary conditions and are sufficiently smooth for integration by parts. The creative step is to determine this finite family suitably. For concreteness, consider the domain of definition for the differential equation to be a nice region in the plane, a polygon perhaps. The region is divided up into pieces. Over each piece, suitable approximating functions are defined. An example is dividing a polygonal region into triangles, followed by piecewise linear interpolation over the triangles. The term "finite elements" refers either to the triangles or to the linear functions. (Which it is, depends on the author.) Of course, piecewise linear interpolation cannot be carried out, because the values of the solution of the differential equation are not known inside the region. But these values are determined by a least squares approximation to the solution, in the inner product corresponding to the variational principle. The Finite Element Method is also called the Rayleigh-Ritz-Galerkin Method, since these are particular cases of it.

Perhaps an example helps: The differential equation is Poisson's equation

$$-\Delta u = f. \tag{1}$$

The approximation is of the form

$$u(x, y) \simeq \sum_{i=1}^{n} A_i B_i(x, y) \tag{2}$$

where the $B_i(x, y)$ are the basis functions and the A_i are numbers to be determined by the Finite Element Method, as follows:

$$\sum_{i=1}^{n} A_i a(B_i, B_j) = (f, B_j), \qquad j = 1, \ldots, n, \tag{3}$$