but stronger than bounded pointwise convergence. A sequence $f_n \in L^\infty$ is said to converge strictly to $f \in L^\infty$ if $f_n \to f$ pointwise and $\Sigma|f_{n+1} - f_n| \in L^\infty$. Strict convergence is stronger than bounded pointwise convergence so any weak * closed subspace of L^∞ is closed under strict convergence. If $S \subseteq L^\infty$ is a weak * closed subspace and Λ is a linear functional on S, Λ is called strictly continuous if whenever $f_n \to f$ strictly then $\Lambda(f_n) \to \Lambda(f)$. It is clear that any linear functional Λ, where $\Lambda(f) = \int f\varphi\, dm$ with $\varphi \in L^1$ is strictly continuous. The proof of the Mooney-Havin theorem now follows from two key facts. (i) If $\{\Lambda_n\}$ is a sequence of strictly continuous linear functionals on a weak * closed subspace $S \subseteq L^\infty$ and if $\Lambda(f) = \lim_{n\to\infty} \Lambda(f_n)$ exists for all $f \in S$ then Λ is strictly continuous; (ii) if $t_n \to 0$, $t_n > 0$, then $f/(1 + t_n h) \to f$ strictly.

There are many other topics covered in these notes that I have not mentioned. For example there is a chapter on imbedding analytic discs and a chapter on rational approximation.

The material is well organized and carefully presented. Many of the proofs are extremely elegant.

PATRICK AHERN

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The theory of partitions, by George E. Andrews, in Encyclopedia of Mathematics and its Applications, volume 2, Addison-Wesley Publishing Company, Advanced Book Program, London, Amsterdam, Don Mills, Ontario, Sydney, and Tokyo, 1976, xiv + 255 pp., $19.50.

The serious study of partitions probably started when Euler was asked how many ways fifty could be written as the sum of seven summands. From this modest beginning a beautiful field has grown up that has connections with a number of different areas of mathematics.

Ferrers, in a letter to Sylvester, observed that it was possible to represent a partition by an array of dots. For example, $7 = 4 + 2 + 1$ is represented by

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A large number of identities can be proved by suitably counting the dots in a Ferrers graph. One beautiful example is F. Franklin's proof of the following result of Euler.

Let $P_n(D, e)$ denote the number of partitions of n into an even number of distinct parts and $P_n(D, o)$ the number of partitions of n into an odd number of distinct parts. Then

$$P_n(D, e) - P_n(D, o) = \begin{cases} 0, & n \neq k(3k \pm 1)/2, \\ (-1)^k, & n = k(3k \pm 1)/2, k = 0, 1, \ldots. \end{cases} \tag{1}$$

This proof is given in Chapter 1 and anyone who is interested in seeing how mathematics can be done without having to introduce many definitions should read this proof. There is an identity equivalent to (1) which was given by Euler. It is

$$\sum_{-\infty}^{\infty} (-1)^k q^{k(3k+1)/2} = \prod_{k=1}^{\infty} (1 - q^k). \tag{2}$$

Combinatorial proofs have been found for many other identities similar to (2). Some recent examples of D. Bressoud [11], [12], [13] are particularly impressive. However it has been impossible (so far) to prove the deepest identities by these methods.

In the nineteenth and some of the twentieth century these identities were thought to live among elliptic functions and their close relations, the modular functions. For example, Euler's identity (2) was generalized by Gauss in an unpublished work [21] and by Jacobi [31] to

$$\sum_{-\infty}^{\infty} q^{n^2} x^n = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + xq^{2n+1})(1 + q^{2n+1}x^{-1}). \tag{3}$$

It took Gauss a number of years to find the easiest proof of (3), [22], but he never published it. Cauchy [16] published the same proof fourteen years after Jacobi published his proof of (3). This proof has been republished often, [7], [28], [29], [33], [34a], [39], but it has also been forgotten or never learned by many people who wrote on this subject. For example, Bellman [9] wrote that there does not seem to be a simple proof of (3). The proof of Gauss and Cauchy is simple. First extend the binomial theorem to

$$\sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} x^k = (q^{-n}x; q)_n \tag{4}$$

where

$$\begin{aligned}(a; q)_k &= (1 - a)(1 - aq) \cdots (1 - aq^{k-1}), \quad & k &= 1, 2, \ldots, \\ &= 1, & k &= 0.\end{aligned} \tag{5}$$

Then take $n = 2m$, shift the summation index by m so that the sum goes from $-m$ to m, simplify, and let $m \to \infty$. It is very easy to prove (4) by functional equations. Formula (3) is harder to obtain directly from functional equations, since it is hard to find the term on the right hand side which is independent of x. It is possible to find it by using (2), and there are other methods, but there are no trivial ones. The reason for this is that there is no value of x which reduces the sum (3) to a trivial sum. In (4) the value $x = 0$ gives $1 = 1$.

The details above were given because this was the first indication that polynomials might play a central role in the study of partition identities. The best proof of this observation comes in a very important series of papers of L. J. Rogers [35], [36], [37], [38]. These papers were completely ignored for over twenty years until Ramanujan read [37] in 1917. Two of the identities in this paper are

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty} \tag{6}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{7}$$

Ramanujan expressed great admiration for this paper of Rogers [25, p. 91] and with good reason; for he had been trying unsuccessfully to prove (6) and (7) for a number of years. After discovering these identities (but not being able to prove them) he communicated them to Hardy. Hardy and others Hardy communicated these identities to, including MacMahon and Perron, also could not prove them. MacMahon realized these identities had a combinatorial interpretation, but no one has yet found a combinatorial proof that gives a bijection between the objects being counted. There are many proofs of (6) and (7), and most of them are closely related to the original proof of Rogers. The idea behind Rogers' proof is important, and still not appreciated by most people who work in this area. Partly this is because Rogers did not completely understand his own work, and so [37] is hard to read, but a more important reason is that a later paper of Rogers [38] has not been read carefully (if it has been read at all). To see what these papers of Rogers contain we must give some background.

Most of the work on special functions in the nineteenth century centered around elliptic functions, and in the last part of the century dealt with modular functions. The results in this field were striking and attracted much interest. There was another set of results that progressed slowly, but in the end it turned out to be more useful. This was the study of hypergeometric series and the extension of them to basic hypergeometric series. A generalized hypergeometric series is a series $\sum_{n=0}^{\infty} a_n$ with a_{n+1}/a_n a rational function of n. Gauss [20] studied the case

$$\frac{a_{n+1}}{a_n} = \frac{(n+a)(n+b)}{(n+c)(n+1)}\, x, \qquad a_0 = 1,$$

and found many important results. Among them are three term recurrence relations and continued fractions that follow from these recurrence relations. Earlier Euler had studied this function and found an integral representation and a differential equation it satisfied. Most of the work on hypergeometric series in the nineteenth century dealt with the differential equation of Euler and extensions of it. One exception was some of Heine's work [26], [27]. He generalized the ideas in Gauss' published paper on hypergeometric functions to a more general class of functions. He considered $\sum_{n=0}^{\infty} a_n$ with

$$\frac{a_{n+1}}{a_n} = \frac{(1-aq^n)(1-bq^n)}{(1-cq^n)(1-q^{n+1})}\, x, \qquad a_0 = 1.$$

When $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$ and $q \to 1$ this reduces to the series Gauss studied.

A generalized basic hypergeometric series $\sum a_n$ has a_{n+1}/a_n as a rational function of q^n. Heine's series

$${}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n}\, x^n,$$

is an example, as is the series in the q-binomial theorem (4). So is the series in (3). For $a_{n+1}/a_n = q^{2n+1}x$ in this case. The terms in the series are labeled from $-\infty$ to ∞ rather than from 0 to ∞, but this is allowed.

Heine found a transformation formula

$$ {}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix} ; q, x\right) = \frac{(b;q)_\infty (ax;q)_\infty}{(c;q)_\infty (x;q)_\infty} \, {}_2\varphi_1\left(\begin{matrix} c/b, x \\ ax \end{matrix} ; q, b\right), $$

some recurrence relations, continued fractions that follow from these recurrence relations, and a number of other important results. Rogers tried to understand some of these results of Heine and was ultimately led to a set of polynomials that generalize the ultraspherical polynomials $C_n^\lambda(x)$. Rogers had a typical education of the time; he knew about spherical harmonics and so had some idea what to look for when considering his polynomials. He found many facts about these polynomials. Two were particularly important. To set notation, define $C_n(x;\beta|q)$ by the recurrence relation

$$ \begin{aligned} (1-q^{n+1})C_{n+1}(x;\beta|q) &= 2x(1-q^n\beta)C_n(x;\beta|q) \\ &\quad - (1-q^{n-1}\beta^2)C_{n-1}(x;\beta|q), \\ C_0(x;\beta|q) = 1, &\qquad C_{-1}(x;\beta|q) = 0. \end{aligned} $$

When $\beta = q^\lambda$, $C_n(x;q^\lambda|q) \to C_n^\lambda(x)$ as $q \to 1$. Rogers found the connection coefficients in

$$ C_n(x;\beta|q) = \sum_{k=0}^{[n/2]} a(k,n) C_{n-2k}(x;\gamma|q). \tag{8} $$

In [37] he only introduced these polynomials when $\beta = 0$, $\beta = q$ and an appropriate limiting case when $\beta \to 1$. The last two cases were just Tchebychef polynomials of the second and first kind respectively, and so are independent of q. The case $\beta = 0$ gives a set of polynomials that generalize the Hermite polynomials. Rogers used (8) for these values of β and γ and another expansion to find and prove (6) and (7). It is now clear that almost all partition identities, and in fact almost all the identities we know for hypergeometric and basic hypergeometric series in one variable, can be obtained easily from orthogonal polynomials. See **[5]** and **[6]** for some examples. Rogers was really the first to make a start on this project.

The second very important result of Rogers was his determination of the coefficients in

$$ C_n(x;\beta|q)C_m(x;\beta|q) = \sum_{k=0}^{\min(m,n)} b(k,m,n) C_{m+n-2k}(x;\beta|q). $$

The case $q = 1$ was rediscovered by Dougall **[18]** in 1919, and the case $q \to -1$ contains a result found by Hylleraas **[30]** in 1962 while studying the Yukawa potential. This gives some indication why Rogers' work was not appreciated. He was far ahead of anyone else, but he did not know quite enough to be able to see what was really happening. The polynomials of

Rogers are orthogonal but the weight function was only found in 1977. See [8] and [42].

The orthogonal polynomial setting for partition identities is not explained in this book, because no one was aware of it when the book was written. However, the recurrence relations, or difference equations, that underlie both orthogonal polynomials in one variable and most partition identities are treated in some detail in a number of chapters.

For over forty years the Rogers-Ramanujan identities (6) and (7) were admired, new proofs were found, and the general feeling was that these identities existed in isolation. There were even theorems that seemed to say that the combinatorial interpretation of the R.-R. identities could not be extended to any modulus other than 5. In 1961 B. Gordon [23] showed that a combinatorial result held for all odd moduli greater than 3. Later Andrews [1] found a way of translating Gordon's results into multisum identities. He then found the multiple basic hypergeometric series identity that was responsible for these identities [2] and he (for $4k + 2$) and Bressoud (for $4k$) finally completed the work of Gordon. There are now Rogers-Ramanujan identities for all moduli. See [14], [15].

This is not the final word on these problems. Macdonald [32] has shown that the Jacobi triple product identity (3) is one of an infinite number of identities that arise from affine root systems of simple Lie algebras. After the first two identities, (3) and the quintuple product identity, the series are all multiple series. While we have a fairly good idea of what can be done for hypergeometric and basic hypergeometric series in one variable (if one ignores the potentially very important results of Gosper [24], some of which do not fit into the existing framework) we have no idea what to expect in several variables. The Macdonald identities, and some conjectures of his that seem even more important than his identities, suggest that a deep theory still exists that we need to discover. There are also indications of this in a number of branches of physics, from nuclear physics [34], angular momentum theory [10] and statistical mechanics [19]. All of these deal with the case $q = 1$, but q extensions exist of many of these results, some as theorems and others as conjectures. In any case Hardy was clearly wrong when he said that the great age of formulas may be over [25, p. 14]. It would be useful if we had someone with Ramanujan's great ability to manipulate formulas to aid us. However, we have one legacy of Ramanujan which may help us a bit. Andrews has found a large number of pages of identities of Ramanujan which are almost surely work from the last year of his life [4]. In working out proofs of these identities we may be led to discover new methods that can be used on the conjectures that we currently cannot prove.

There are other sides to the study of partitions. One aspect of partitions is related to modular functions. From these functions come asymptotic formulas for the unrestricted partition function and some congruences for this function. Both of these results started with conjectures of Ramanujan and he obtained some of the first results. The Hardy-Ramanujan asymptotic results were completed by Rademacher, and Ramanujan's congruence conjectures were proved by Watson and Atkin. The asymptotic results have been partly extended to another class of functions, the mock theta functions that

Ramanujan discovered in his last year of life, and so they are not exclusively restricted to modular functions. The congruence theorems have a minor combinatorial flavor which was empirically discovered by Dyson and proved by Atkin and Swinnerton-Dyer, but even this seems to be controlled by modular functions. Other recent results of great interest dealing with modular functions are those of Deligne [17] and Serre and Swinnerton-Dyer [41].

One further aspect of partitions is intimately tied up with group theory. Part of the connection comes from plane partitions and their connection with representations of the symmetric group, but there are other connections. These include a conjecture of Lusztig and Macdonald which was proved by Andrews [3], Macdonald's beautiful group theoretic proofs of MacMahon's generating function for plane partitions and MacMahon's conjectured generating function for symmetric plane partitions, and a conjecture of Macdonald for plane partitions with a further rotational symmetry. All of Macdonald's work is still unpublished.

A further connection with group theory comes from the spherical functions on some discrete two point homogeneous spaces with finite classical groups acting on the spaces. The spherical functions are orthogonal polynomials which can be given as basic hypergeometric series. For a survey of this see Stanton [40]. Further references are given there.

There are other applications mentioned in the book under review. Most readers will have very little trouble finding one that is close to their interests. This book is a good introduction to a fascinating part of mathematics. Not all topics are treated in depth, but references are provided for the reader who wishes to study the subject further. With the exception of the unattractive look of some of the Ferrers graphs, and more misprints than there should be, the book is a very pleasant one to read.

REFERENCES

1. G. Andrews, *An analytic generalization of the Rogers-Ramanujan identities for odd moduli*, Proc. Nat. Acad. Sci U.S.A. **71** (1974), 4082–4085.

2. ______, *Problems and prospects for basic hypergeometric functions*, Theory and Application of Special Functions (R. Askey, ed.), Academic Press, New York, 1975, pp. 191–224.

3. ______, *q-series and the Lusztig-Macdonald-Wall conjectures*, Invent. Math. **41** (1977), 91–102.

4. ______, *An introduction to Ramanujan's "lost notebook"*, Amer. Math. Monthly (to appear).

5. ______, *Connection coefficient problems and partitions*, Proc. Sympos. Pure Math., vol. 34, Amer. Math. Soc., Providence, R. I., 1978, pp. 1–24.

6. G. Andrews and R. Askey, *Enumeration of partitions: The role of Eulerian series and q-orthogonal polynomials*, Higher Combinatorics (M. Aigner, ed.), Reidel, Dordrecht, Holland, 1977, pp. 3–26.

7. P. Appell and E. Lacour, *Principes de la théories des fonctions elliptiques*, Gauthier-Villars, Paris, 1897.

8. R. Askey and M. E.-H. Ismail, *A generalization of ultraspherical polynomials* (to appear in a book dedicated to P. Turán's memory).

9. R. Bellman, *A brief introduction to theta functions*, Holt, Rinehart and Winston, New York, 1961.

10. L. Biedenharn and J. Louck, *Angular momentum in quantum physics—Theory and application* (to appear).

11. D. Bressoud, *A new family of partition identities*, Pacific J. Math. **77** (1978).

12. ______, *Combinatorial proof of Schur's theorem* (submitted).

13. ______, *Extension of the partition sieve* (submitted).

14. ______, *A generalization of the Rogers-Ramanujan identities for all moduli*, J. Combinatorial Theory Ser. A (to appear).

15. ______, *A functional generalization of the Rogers-Ramanujan identities with interpretation* (submitted).

16. A. Cauchy, *Second Mémoire sur les fonctions dont plusieurs valeurs sont liées entre par une équation linéaire, Oeuvres*, 1re Série, Tome VIII, 50–55, Gauthier-Villars, Paris, 1893, reprinted from C. R. 1843.

17. P. Deligne, *La conjecture de Weil*. I, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307. MR **49** #5013.

18. J. Dougall, *A theorem of Sonine in Bessel functions, with two extensions to spherical harmonics*, Proc. Edinburgh Math. Soc. **37** (1919), 33–47.

19. F. Dyson, *Statistical theory of the energy level of complex systems*. I, J. Math. Phys. **3** (1962), 140–156.

20. C. F. Gauss, *Disquisitiones generales circa seriem infinitam* $1 + \alpha\beta x/1 \cdot \gamma + \alpha(\alpha+1)\beta(\beta+1)xx/1\cdot 2\,\gamma(\gamma+1) + \cdots$, *Werke*, vol. 3, Königlichen Gesellschaft der Wissenschaften, Göttingen, 1866, pp. 123–162 (originally published in 1813).

21. ______, *Zur Theorie der neuen Transscendenten*. II, *Werke*, vol. 3, Göttingen, 1866, pp. 436-445.

22. ______, *Hundert Theoreme über die neuen Transscendenten*, *Werke*, vol. 3, Göttingen, 1866, pp. 461–469.

23. B. Gordon, *A combinatorial generalization of the Rogers-Ramanujan identities*, Amer. J. Math. **83** (1961), 393–399.

24. R. W. Gosper, *A calculus of series rearrangements*, Algorithms and Complexity, New Directions and Recent Results (J. Traub, ed.), Academic Press, New York, 1976, pp. 121–151.

25. G. H. Hardy, *Ramanujan*, Cambridge Univ. Press, London, 1940; reprinted by Chelsea, New York, 1959.

26. E. Heine, *Untersuchungen über die Reihe*

$$1 + \frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}x + \frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{2})(1-q^{\gamma})(1-q^{\gamma+1})}x^2 + \cdots,$$

J. Reine Angew Math. **34** (1845), 285–328.

27. ______, *Theorie der Kugelfunctionen und der verwandten Functionen*, Reimer, Berlin, 1878.

28. P. Henrici, *Applied and computational complex analysis*, vol. 2, Wiley-Interscience, New York, 1977.

29. C. Hermite, *Oeuvres*, Tome II, Gauthier-Villars, Paris, 1908, pp. 153–156; reprinted from *Calcul différentiel et Calcul intégral de Lacroix*, 6^{e} edition, Paris, 1862.

30. E. Hylleraas, *Linearization of products of Jacobi polynomials*, Math. Scand. **10** (1962), 189–200.

31. C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Regiomontis, fratrum Borntraeger 1829; reprinted in *Gesammelte Werke*, Volume 1, Reimer, Berlin, 1881; reprinted by Chelsea, New York, 1969, pp. 49–239.

32. I. G. Macdonald, *Affine root systems and Dedekind's η-function*, Invent. Math. **15** (1972), 91–143.

33. G. Pólya and G. Szegö, *Problems and theorems in analysis*, vol. I, Springer-Verlag, New York, Heidelberg, Berlin, 1972.

34. G. Racah, *Theory of complex spectra*. I, II, III, IV, Phys. Rev. **61** (1942), 186–197; **62** (1942), 438–462; **63** (1943), 367–382; **76** (1949), 1352–1365; reprinted in *Quantum theory of angular momentum* (L. Biedenharn and H. VanDam, ed.), Academic Press, New York, 1965.

34a. O. Rausenberger, *Lehrbuch der Theorie der Periodischen Functionen einer Variabeln*, Teubner, Leipzig, 1884.

35. L. J. Rogers, *On a three-fold symmetry in the elements of Heine's series*, Proc. London Math. Soc. **24** (1893), 171–179.

36. ______, *On the expansion of some infinite products*, Proc. London Math. Soc. **24** (1893), 337–352.

37. ______, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.

38. ______, *Third memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **26** (1895), 15–32.

39. K. H. Schellbach, *Die Lehre von den Elliptischen Integralen und den Theta-Functionen*, Reimer, Berlin, 1864.

40. D. Stanton, *Some basic hypergeometric polynomials arising from finite classical groups*, Ph.D. thesis, Univ. of Wisconsin, Madison, 1977.

41. H. P. F. Swinnerton-Dyer, *On l-adic representations and congruences for coefficients of modular forms* (II), Modular Functions in One Variable V (J.-P. Serre and D. B. Zagier, eds.), Springer-Verlag, Berlin and New York, 1977.

42. J. Wilson, Ph.D. thesis, Univ. of Wisconsin, Madison, 1978.

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Nonlinearity and functional analysis, Melvyn S. Berger, Academic Press, New York, San Francisco, London, 1977, xix + 415 pp., $24.50.

Linear functional analysis evolved as the natural gathering point for a number of different investigations into the solvability of linear equations which were either in the form of integral equations or in the form of countable systems of linear scalar equations in which the unknown was a sequence of numbers. As the subject developed much broader areas of applicability became evident. These applications, in turn, spawned further abstract development, and the abstract results themselves assumed an intrinsic interest. The basic approach of functional analysis, where one considers functions to be points in a large space of related functions and lets the differential or integral operator act on these points, has also been very successful in treating nonlinear problems.

The division between the linear and the nonlinear theory is, of course, not so sharp. As we know from calculus, a great deal of information about a system of nonlinear equations is obtained from their local linear approximation. Moreover, it is often possible to glean information about a linear problem by considering a related nonlinear problem; this is so strikingly demonstrated in Lomonosov's recent results on the invariant subspace problem for linear operators [5].

Berger's aim is to give a systematic treatment of some of the fundamental abstract nonlinear results and of their application to certain concrete problems in geometry and physics.

The study of nonlinear operators acting on infinite dimensional spaces has an obvious starting point–study the finite-dimensional case, a finite system of scalar equations in a finite number of unknowns. Even at this step we note that a fairly complete description of the solutions would be very difficult; when $p(x)$ is a polynomial in a single variable the study of solutions of $p(x) = 0$ is the subject matter of classical algebraic geometry.

Let n and k be integers, $n \geq 1$, $k + n \geq 1$ and let $\mathcal{O}$ be a bounded open subset of $\mathbf{R}^n$. Suppose $f\colon \bar{\mathcal{O}} \to \mathbf{R}^{n+k}$ is continuous.

If f is one-to-one then $k \geq 0$. Moreover, assuming f is one-to-one, $f(\mathcal{O})$ is open in $\mathbf{R}^{n+k}$ iff $k = 0$. (F1)