BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 6, November 1978

G-FOLIATIONS AND THEIR CHARACTERISITC CLASSES[1]

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Introduction. Simple examples of foliations arise from submersions. Let M^n and N^q be smooth manifolds of dimensions n and q respectively, and let f: $M \to N$ be a smooth submersion, i.e. rank $(df_x) = q < n$ for all $x \in M$. Then the partition of M by the connected components of the inverse images $f^{-1}(y)$ for $y \in N$ defines a foliation of M. If the target manifold is further equipped with a G-structure in the sense of Chern [CH], where G is a closed subgroup of $GL(q)$, then the foliation of M by the components of the inverse images of the submersion f is an example of a G-foliation. Since foliations are at least locally defined by submersions as explained above, we can think of them as *relative* manifolds. In this view G-foliations are then the corresponding *relative* G-structures. This concept embraces Riemannian, conformal, symplectic, almost complex foliations, etc. In short: the classical geometry of G-structures has its relative counterpart in the geometry of G-foliations. Much progress has been made in this theory in the past half dozen years through the work of Bernstein-Rosenfeld, Bott-Haefliger, Chern-Simons, Gelfand-Fuks, Godbillon-Vey, Kamber-Tondeur, Heitsch, Thurston and many others. In this lecture we discuss selected topics in the theory of characteristic classes which are naturally attached to G-foliations. This theory is very much in flux and the present exposition is by no means a survey of even this limited field. The aim has rather been to supply a rich variety of examples together with the necessary conceptual and computational background, so as to show the attractiveness of the subject.

1. G-foliations and foliated bundles. For surveys on the general theory of foliations we refer to Lawson [L1], [L2]. Let M be a smooth manifold. An

An expanded version of an invited address delivered by the second author before the Meeting of the Society in Ann Arbor, Michigan on November 6, 1976; received by the editors September 23, 1977.

AMS (MOS) subject classifications (1970). Primary 57D20; Secondary 57D30.

[1]This work was partially supported by grants from the National Science Foundation.

infinitesimal description of a smooth foliation $\mathcal{F}$ on M is given by the vectors tangent to the foliation. They form a subbundle $L \subset T_M$ of the tangent bundle of M which is involutive, i.e. for any two vectorfields $X \in \Gamma L$, $Y \in \Gamma L$ (X and Y are (local) sections of L), the bracket $[X, Y] \in \Gamma L$. The theorem of Frobenius states that any such involutive subbundle $L \subset T_M$ does indeed occur as the bundles of vectors tangent to a well-defined foliation, namely the foliation of M by the integral leaves of L.

The normal bundle Q is the quotient defined by the short exact bundle sequence

$$0 \to L \to T_M \xrightarrow{p} Q \to 0. \tag{1.1}$$

The codimension q of the foliation is the (fiber) dimension of Q. In the case of a foliation defined by the connected fibers of a submersion f: $M \to N$ as explained in the introduction, the normal bundle Q is the pull-back $f^* T_N$ of the tangent bundle of the target space ($q = \dim N$). It is then visibly a trivial bundle when restricted to any leaf. This phenomenon is reflected in the general case by the flatness of Q when restricted to any leaf. The flatness of a smooth bundle is characterized by the existence of a curvature free connection on the bundle. In this vein the partial flatness of Q (flatness when restricted to any leaf) is characterized by the existence of a connection which is curvature free along the leaves. This is made precise by the Bott connection in Q, defined by

$$\nabla_X s = p[X, Y] \quad \text{for} \quad X \in \Gamma L. \tag{1.2}$$

Here s denotes a section of Q and Y a vectorfield projecting to s under p (see (1.1)). The covariant derivative of s is canonically defined only for $X \in \Gamma L$, i.e. in directions tangent to the foliation. Thus ∇ is a partial connection in the sense that it is only defined for certain vectorfields X, but otherwise has all the usual formal properties of a connection (see (2.10) in **[KT 9]**). The flatness of this partial connection is characterized by the identity

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = 0 \tag{1.3}$$

for X, $Y \in \Gamma L$.

1.4 DEFINITION. An *adapted* connection in Q is any (ordinary) covariant derivative operator in Q, which for $X \in \Gamma L$ reduces to the definition (1.2). While the partial connection given by (1.2) is canonically given, an adapted connection involves a choice (the existence presents no problem in this smooth situation). Thus e.g. the curvature $R(X, Y)$ defined by the left-hand side of (1.3) has no intrinsic meaning for arbitrary vector fields X, Y on M. Its vanishing for X, $Y \in \Gamma L$ on the other hand is intrinsic. In the same vein the holonomy of an adapted connection has no intrinsic meaning. What does have an intrinsic meaning is the holonomy of the restriction of Q to any leaf F of the foliation. Since Q/F is flat, this holonomy is characterized by a representation h: $\pi_1(F) \to GL(q)$, the holonomy homomorphism (see e.g. **[KT 1**, p. 10**]**).

It is convenient to consider the same situation also from the principal bundle point of view. An adapted connection in Q corresponds then to a connection 1-form ω in the $GL(q)$-frame bundle $F(Q)$ of Q. The horizontal spaces $H_u = \ker \omega_u$ for $u \in F(Q)$ define a subbundle $H \subset T_{F(Q)}$. If π: $F(Q) \to M$ denotes the projection, then only the subbundle $\tilde{L} \subset T_{F(Q)}$ defined by $\tilde{L}_u = H_u \cap \pi^{-1}_*(L_x)$ for $x = \pi(u)$ has intrinsic meaning. The identity (1.3) translates to the involutivity of $\tilde{L}$. Thus the intrinsic structure in $F(Q)$ is a canonical foliated bundle structure in the following sense.

1.5 DEFINITION. Let P be a principal G-bundle with projection π: $P \to M$. P is a *foliated bundle* if there is a G-equivariant foliation $\tilde{L} \subset T_P$ such that for each $u \in P$ the intersection $\tilde{L}_u \cap G_u$ with the tangent space G_u to the fiber of P through u is the 0-space.

The G-equivariance of $\tilde{L}$ gives rise to a quotient foliation $L = \tilde{L}/G$ on M. An adapted connection in a foliated G-bundle is then a connection 1-form ω such that $\ker \omega_u \supset \tilde{L}_u$ for each $u \in P$. This notion again involves a choice while the existence poses no problem. The intrinsic structure is also called a partial flat connection in P (partial because the direct sum $G_u \oplus \tilde{L}_u$ does in general not equal T_uP). All this is elaborated in great detail in **[KT 9]**. These concepts were introduced independently by the authors and Molino.

Let now G be a closed subgroup of $GL(q)$.

1.6 DEFINITION. A codimension q foliation $L \subset T_M$ on a manifold M is a *G-foliation*, if there exists a G-reduction P of the $GL(q)$-frame bundle $F(Q)$, such that the canonical foliated bundle structure on $F(Q)$ arises from a foliated bundle structure on P.

Recall that P is a G-reduction of $F(Q)$, if $F(Q)$ is the natural extension of P under the change of groups from G to $GL(q)$, i.e.

$$F(Q) \cong P \times_G GL(q). \tag{1.7}$$

A G-equivariant foliation on P gives rise to a $GL(q)$-equivariant foliation on $F(Q)$. The requirement in definition (1.6) is then that the canonical foliation $\tilde{L}$ on $F(Q)$ arises in this fashion from a foliated bundle structure on P. Note that both foliations on P and $F(Q)$ project onto the given foliation on M. In terms of adapted connections, this condition means that there is an adapted connection with holonomy group in G.

For the previously discussed example of a foliation defined by a submersion f: $M \to N$, the pullback $f^*\bar{P} = P$ of a G-structure $\bar{P}$ on N has obviously the desired properties with respect to the bundle $F(Q) \cong f^*F(N)$, where $F(N)$ denotes the $GL(q)$-frame bundle of N.

To discuss alternate definitions of G-foliations, consider a (local) vectorfield $X \in \Gamma L$. Any such vectorfield has a canonical lift to a vectorfield $\tilde{X} \in \Gamma\tilde{L}$ on $F(Q)$, its partial horizontal lift **[KT 9**, p. 14**]**, characterized by $\pi_*\tilde{X}_u = X_x$ for $u \in F(Q)$, $\pi(u) = x$. Recall further that a G-reduction P of $F(Q)$ is given by a section s: $M \to F(Q)/G$ of the projection $F(Q)/G \to M$ in the form $P \cong s^*F(Q)$, the pullback of the G-bundle $F(Q) \to F(Q)/G$ under s.

At this point we need to recall Haefliger's cocycle definition of a foliation as follows. A codimension q foliation on M is given by an open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ and submersions f_i: $U_i \to \mathbf{R}^q$ for $i \in I$, satisfying the following properties. For each $i, j \in I$ and $x \in U_i \cap U_j$ there is a local diffeomorphism γ_{ij}^x of $\mathbf{R}^q$ such that $f_j = \gamma_{ji}^x f_i$ on a neighborhood of x. The cocycle condition $\gamma_{ki}^x = \gamma_{kj}^x \circ \gamma_{ji}^x$ guarantees that the local foliations on the U_i defined by the submersions f_i piece together to a global foliation. If now $\mathbf{R}^q$ is equipped with a G-structure, and the local diffeomorphisms γ_{ij}^x are local automorphisms of this G-structure, then the corresponding foliation on M is a G-foliation. $\mathbf{R}^q$ of course has a particular canonical flat G-structure. G-foliations defined by a Haefliger cocycle with respect to the local automorphisms of this flat G-structure are integrable. But every G-foliation (integrable or not) can be defined by a Haefliger cocycle with respect to the local isomorphisms of some G-manifold.

Such a G-manifold is constructed as follows (Duchamp [**D**, 2.4]). Let $F(\mathbf{R}^q) \to \mathbf{R}^q$ be the $GL(q)$-frame bundle of $\mathbf{R}^q$ and consider the total space $N(G, \mathbf{R}^q)$ of germs of C^∞-sections of the projection $F(\mathbf{R}^q)/G \to \mathbf{R}^q$. Then $N(G, \mathbf{R}^q)$ is equipped with a canonical G-structure.

The following characterizations of G-foliations illustrate then the concept from various points of view.

1.8 PROPOSITION [**D**]. *Let $\mathfrak{F}$ be a foliation of codimension q on M, $G \subset GL(q)$ a closed subgroup and P a G-reduction of the frame bundle $F(Q)$. The following conditions are equivalent.*

(i) *The canonical foliated bundle structure on $F(Q)$ arises from a foliated bundle structure of P.*

(ii) *There is an adapted connection in $F(Q)$ with holonomy group in G.*

(iii) *The section s: $M \to F(Q)/G$ maps leaves of $\mathfrak{F}$ onto leaves of the quotient foliation $\tilde{L}/G$ on $F(Q)/G$.*

(iv) *For every vectorfield X on M tangent to $\mathfrak{F}$ the flow of the partially horizontal lift $\tilde{X}$ on $F(Q)$ leaves P invariant.*

(v) *For every $x \in M$ there is an open neighborhood $U \subset M$, a q-dimensional manifold N with a G-structure $\bar{P}$ and a submersion f: $U \to N$ with connected fibers such that the restriction of $\mathfrak{F}$ to U is defined by the fibers of f and such that $P/U \cong f^*\bar{P}$. Moreover, the transition functions γ_{ij}^x for each pair f_i, f_j of submersions as above are local automorphisms of the G-structure $\bar{P}$ on N.*

We explain one more important geometric concept. Let $L \subset T_M$ be a G-foliation on M, and ω the connection form of an adapted connection in the bundle of G-frames P.

1.9 DEFINITION. An adapted connection is *basic*, if

$$\Theta(\tilde{X})\omega = 0 \quad \text{for all } X \in \Gamma L.$$

Here $\tilde{X} \in \Gamma\tilde{L}$ denotes the partially horizontal lift to P of a vectorfield $X \in \Gamma L$ and $\Theta(\tilde{X})$ the Lie derivative along $\tilde{X}$.

If the foliation is in particular defined by a submersion f: $M \to N$ onto a manifold N with G-structure $\bar{P}$, than any connection $\bar{\omega}$ in the bundle $\bar{P}$ pulls back to a basic connection ω in $P = f^*\bar{P}$, which explains the terminology. We have however to point out that this terminology is not universally accepted. In fact some authors use the adjective basic for the adapted connections in the sense of Definition 1.4. Molino's terminology for these connections is projectable, which evokes the same associations as basic.

In terms of the cocycle description, the existence of a basic connection means that the defining pseudogroup consists of local automorphisms of a G-structure preserving a connection in that G-structure. The basic connection is locally given as the pullback of that connection via the defining submersions. Since the transition functions are connection preserving, the connection in P is well defined.

In the examples of §2, we make use of the following calculus for foliations. Let ω be a covariant tensor of degree p on Q. Then the Lie derivative $\Theta(X)\omega$ for $X \in \Gamma L$ is given by the formula

$$(\Theta(X)\omega)(s_1, \ldots, s_p) = X\omega(s_1, \ldots, s_p) - \sum_{i=1}^{p} \omega(s_1, \ldots, \nabla_X s_i, \ldots, s_p) \tag{1.10}$$

for $s_1, \ldots, s_p \in \Gamma Q$ and $\nabla_X s_i$ as in (1.2). For $p = 1$ this formula shows that in fact

$$(\Theta(X)\omega)(s) = X\omega(s) - \omega(\nabla_X s) \equiv (\nabla_X^* \omega)(s) \tag{1.11}$$

i.e. the dual connection ∇_X^* in Q^* is given by the Lie derivative. The identity $\Theta(X) = i(X)d + di(X)$ implies in view of $i(X)\omega = 0$ for $X \in \Gamma L$ also the formula

$$\nabla_X^* \omega = i(X)d\omega \quad \text{for} \quad \omega \in \Gamma Q^*. \tag{1.12}$$

We further need in §2 the Lie derivative of a tensor J on Q which is contravariant of degree 1, covariant of degree p. It is given for $X \in \Gamma L$ by the formula

$$(\Theta(X)J)(s_1, \ldots, s_p) = \nabla_X J(s_1, \ldots, s_p) - \sum_{i=1}^{p} J(s_1, \ldots, \nabla_X s_i, \ldots, s_p) \tag{1.13}$$

where $s_1, \ldots, s_p \in \Gamma Q$.

2. Examples.

2.1 RIEMANNIAN FOLIATIONS. Here the group $G = O(q)$. These are the foliations with bundle-like metrics introduced by Reinhart [**RE**]. The normal bundle Q is equipped with a metric g (fiber metric) such that

$$\Theta(X)g = 0 \quad \text{for all} \quad X \in \Gamma L. \tag{2.2}$$

Here $\Theta(X)g$ denotes the Lie derivative of g. This condition is by (1.10) equivalent to the identity

$$Xg(s, t) = g(\nabla_X s, t) + g(s, \nabla_X t) \tag{2.3}$$

for $X \in \Gamma L$ and $s, t \in \Gamma Q$. An equivalent definition of a Riemannian foliation is the condition that the transition functions in the cocycle definition can be chosen as local isometries of a Riemannian manifold. The metric g is locally the pullback of the Riemannian metric on the target via the local submersions. A Riemannian foliation obviously admits a basic connection, namely the local pullbacks of the Riemannian connection on the target via the local submersions (Pasternack [**P**]). If the model manifold is in particular $\mathbf{R}^q$ with its Euclidean metric, the foliation is an integrable Riemannian foliation.

2.4 $SL(q)$-FOLIATIONS. In this case the natural bundle Q is equipped with a nondegenerate q-form $\nu \in \Gamma\Lambda^q Q^*$ (fiber volume) such that

$$\Theta(X)\nu = 0 \quad \text{for all} \quad X \in \Gamma L. \tag{2.5}$$

By (1.10) this condition is equivalent to the identity

$$X\nu(s_1, \ldots, s_q) = \sum_{i=1}^{q} \nu(s_1, \ldots, \nabla_X s_i, \ldots, s_q)$$

for $X \in \Gamma L$ and $s_1, \ldots, s_q \in \Gamma Q$. An equivalent definition is that the transition functions in the cocycle description can be chosen as local diffeomorphisms preserving a volume on the model manifold.

2.6 $SO(q)$-FOLIATIONS. These are oriented Riemannian foliations. They are Riemannian foliations which are simultaneously $SL(q)$-foliations.

2.7 CONFORMAL FOLIATIONS. The conformal group $C(q) = O(q) \times \mathbf{R}^*$ is a subgroup of $GL(q)$ via the map $(A, \lambda) \to \lambda A$. A conformal foliation is a $C(q)$-foliation. Examples of conformal foliations are e.g. given in [NS] and [Y].

2.8 SPIN-FOLIATIONS. Here $G = \mathrm{Spin}(n)$. Thus it is convenient to allow $G \to GL(q)$ to be any homomorphism (not necessarily an inclusion) in the definition of a G-foliation.

2.9 ALMOST SYMPLECTIC FOLIATIONS. These are G-foliations of codimension $2q$, where $G = Sp(q) \subset GL(2q)$. The normal bundle Q is equipped with a 2-form $\omega \in \Gamma\Lambda^2 Q^*$ satisfying

$$\omega^q \neq 0, \quad \omega^{q+1} = 0 \quad \text{(nondegeneracy)}, \tag{2.10}$$

$$\Theta(X)\omega = 0 \quad \text{for all} \quad X \in \Gamma L. \tag{2.11}$$

By (1.10) condition (2.11) is equivalent to the identity

$$X\omega(s, t) = \omega(\nabla_X s, t) + \omega(s, \nabla_X t) \tag{2.12}$$

for $X \in \Gamma L$ and $s, t \in \Gamma Q$. An equivalent definition is that the transition functions in the cocycle description can be chosen as local automorphisms of an almost symplectic manifold. The integrability of an almost symplectic foliation is characterized by $d\omega = 0$ [D].

2.13 ALMOST COMPLEX FOLIATIONS. These are G-foliations of codimension $2q$, where $G = GL(q, \mathbf{C}) \subset GL(2q)$. The normal bundle Q is equipped with an almost complex structure $J\colon Q \to Q$ satisfying

$$\Theta(X)J = 0 \quad \text{for all} \quad X \in \Gamma L. \tag{2.14}$$

By (1.13) this condition is equivalent to the identity

$$\nabla_X Js = J\nabla_X s \quad \text{for} \quad X \in \Gamma L, s \in \Gamma Q. \tag{2.15}$$

An equivalent definition is that the transition functions in the cocycle description can be chosen as local automorphisms of an almost complex manifold.

The definition of the Eckmann-Frölicher-Nijenhuis tensor still makes sense in the following relative form. For sections s and t of Q, vectorfields X and Y such that $p(X) = s$, $p(Y) = t$, and $\sigma\colon Q \to T_M$ a section of the canonical projection $p\colon T_M \to Q$, let

$$N(s, t) = 2p\{[\sigma Js, \sigma Jt] - [X, Y]\} - 2Jp\{[X, \sigma Jt] + [\sigma Js, Y]\}. \tag{2.16}$$

This definition is independent of the choice of σ, and the vanishing of N is equivalent to the integrability of the almost complex foliation. Such a foliation has a complex structure only in the normal direction.

2.17 $\{e\}$-FOLIATIONS. An $\{e\}$-reduction of $F(Q)$ is a trivialization of Q. Let $s_1, \ldots, s_q$ be a global frame of Q. L is an $\{e\}$-foliation with respect to the frame $s_1, \ldots, s_q$ if

$$\nabla_X s_i = 0 \quad \text{for all} \quad X \in \Gamma L; \, i = 1, \ldots, q. \tag{2.18}$$

See e.g. Conlon [CL] for the geometric study of such foliations for $q = 2$.

2.19 HOMOGENEOUS G-FOLIATIONS [KT 9]–[KT 12]. Let $\bar{G}$ be a Lie group and $G \subset \bar{G}$ a Lie subgroup. The foliation of $\bar{G}$ by the (left) cosets of G defines a G-foliation with trivial normal bundle Q_G (trivialized by the (right) G-action on $\bar{G}$). The G-reduction P of the frame bundle of Q_G in this case is the trivial G-bundle $P = \bar{G} \times G$ on $\bar{G}$. But the canonical foliated bundle structure on P is not compatible with this trivialization. Namely consider the diagonal G-action on P defined by

$$(\bar{g}, g) \cdot g' = (\bar{g}g', g'^{-1}g) \tag{2.20}$$

for $(\bar{g}, g) \in \bar{G} \times G$ and $g' \in G$. The G-orbits of this free action define a foliation on $\bar{G} \times G$, which under the projection $\bar{G} \times G \to \bar{G}$ maps onto the left coset foliation of $\bar{G}$. The bundle Q_G is associated to P via the adjoint action of G on the quotient $\bar{\mathfrak{g}}/\mathfrak{g}$ of the Lie algebras of $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ respectively, i.e.

$$Q_G \cong P \times_G \bar{\mathfrak{g}}/\mathfrak{g}.$$

The Bott connection in Q_G is induced from the foliated structure on P as described (see [KT 11] for more details).

More generally, let $H \subset G$ be a subgroup which is closed in $\bar{G}$. Then the left coset foliation of $\bar{G}$ by G induces a G-foliation on the homogeneous space $\bar{G}/H$, a homogeneous foliation. The normal bundle of this foliation is associated to the G-bundle

$$\bar{G} \times_H G \to \bar{G}/H. \tag{2.21}$$

The canonical foliated bundle structure described above is H-equivariant and passes to this quotient situation.

Let $\Gamma \subset \bar{G}$ in addition be a discrete subgroup operating properly discontinuously and without fixed points on $\bar{G}/H$, so that the double coset space $\Gamma \backslash \bar{G}/H$ is a manifold. Then the $\bar{G}$-invariant homogeneous G-foliation induced by G on $\bar{G}/H$ passes to a G-foliation on $\Gamma \backslash \bar{G}/H$, a locally homogeneous foliation. The normal bundle of this foliation is associated to the foliated G-bundle

$$(\Gamma \backslash \bar{G}) \times_H G \to \Gamma \backslash \bar{G}/H. \tag{2.22}$$

2.23 QUOTIENT FOLIATIONS [KT 9]–[KT 12]. Let $\bar{P} \xrightarrow{\bar{\pi}} X$ be a foliated $\bar{G}$-bundle. The foliation $L \subset T_{\bar{P}}$ defines the foliation $L_{\bar{G}} = L/\bar{G}$ on $X = \bar{P}/\bar{G}$. For a closed subgroup $G \subset \bar{G}$ there is then defined on $M = \bar{P}/G$ a quotient foliation $L_G = L/G \subset T_M$. The projection $\bar{P} \to \bar{P}/G$ is itself a foliated

G-bundle which we denote $P \xrightarrow{\pi} M$. Thus there is a factorization

$$\begin{tikzcd} \bar{P} = P \arrow[rr, "\bar{\pi}"] \arrow[dr, "\pi"'] & & X = \bar{P}/\bar{G} \\ & M = \bar{P}/G \arrow[ur, "\hat{\pi}"'] & \end{tikzcd} \tag{2.24}$$

of the foliated bundle map $\bar{\pi}$ into the foliated bundle map π and the $\bar{G}/G$-fibration $\hat{\pi}$. Let Q, $Q_{\bar{G}}$ and Q_G denote the normal bundles of the foliations L, $L_{\bar{G}}$ and L_G respectively. It is then e.g. clear that Q_G is an extension of $Q_{\bar{G}}$ (lifted to M) by the bundle tangent to the fibers of $\hat{\pi}$ (see **[KT 12**, (2.7)**]**).

To interpret the foliation on the normal bundle Q of a foliation on X in this fashion, consider the affine frame bundle $A(Q) = F(Q) \times_{GL(q)} A(q)$. The affine group $A(q)$ is the semidirect product $GL(q) \times \mathbf{R}^q$ and $A(Q)$ is equipped with a canonical foliated structure. The factorization corresponding to diagram (2.24) reads in this case

$$\begin{tikzcd} A(Q) \arrow[rr, "\bar{\pi}"] \arrow[dr, "\pi"'] & & X = A(Q)/A(q) \\ & Q \cong A(Q)/GL(q) \arrow[ur, "\hat{\pi}"'] & \end{tikzcd} \tag{2.25}$$

The quotient foliation on Q corresponds to the leaves defined by the Bott connection of Q.

Another example of interest is the following. Consider an oriented Riemannian foliation on X with orthogonal frame bundle P. In the factorization

$$\begin{tikzcd} P \arrow[rr, "\bar{\pi}"] \arrow[dr, "\pi"'] & & X = P/SO(q) \\ & M = P/SO(q-1) \arrow[ur, "\hat{\pi}"'] & \end{tikzcd} \tag{2.26}$$

the fibration $\hat{\pi}$ is the sphere bundle of the normal bundle Q with fiber $SO(q)/SO(q-1) \cong S^{q-1}$.

Returning to the general situation, it is of particular interest when the normal bundle $Q_{\bar{G}}$ of the foliation on X is the zero bundle. This occurs precisely when the original foliation L on $\bar{P}$ projects under $\bar{\pi}$ onto T_X. This means that $\bar{P} \xrightarrow{\bar{\pi}} X$ is a flat bundle, the flat structure being given by the foliation L transverse to the fibers. Thus in this case the quotient foliation is transverse to the fibers of $\hat{\pi}$, which is therefore a flat fiber bundle. The consideration of such a flat fiber bundle $M \xrightarrow{\hat{\pi}} X$ is underlying many of the results in **[KT 10]**–**[KT 12]**. If $F = \bar{G}/G$ denotes the fiber of $\hat{\pi}$, then the flat structure is described by the isomorphism

$$M \cong \tilde{X} \times_\Gamma F, \tag{2.27}$$

where $\tilde{X}$ denotes the universal covering of X and $\Gamma = \pi_1(X)$ acts on F via a homomorphism $\Gamma \to \overline{G}$. It is clear that this situation can be generalized (and it is useful to do so), by replacing $\overline{G}$ by *a* or *the* diffeomorphism group of the fiber. We will return to such situations at the end of this paper.

3. Characteristic classes. The characteristic classes discussed here are all defined over the real or complex numbers. The basic underlying observation is that the Chern-Weil theory of characteristic classes extends in a functorial way to foliated bundles. Applied to the bundle of G-frames, this construction produces in particular characteristic classes for G-foliations. The reasons for viewing these invariants in the larger context of foliated bundles are the following: (i) only in the larger context are these constructions functorial (under change of groups, spaces and foliations), (ii) in the larger context many more examples are incorporated, which do not fit into the narrower framework of foliations.

To illustrate the second point, consider the following extreme cases of foliated bundles. If the foliation $L = T_M$, i.e. the foliation of the base space is the trivial one-leaf foliation of M, then the Definition 1.5 of a foliated bundle reduces to the definition of a flat bundle (with flat bundle structure $\tilde{L} \subset T_P$). Our characteristic class construction produces then invariants for flat bundles. The other extreme situation is the case where the foliation of M is the trivial (0-dimensional) point foliation of M, i.e. $L = 0$. The foliation $\tilde{L}$ on P is then necessarily also the point foliation of P, and the foliated bundle definition reduces to the ordinary (G-bundle) definition. In this case the characteristic class construction is the usual Chern-Weil construction. Thus the point of the foliated bundle definition is seen to be the fact that the foliation on the total space is an extra geometric structure, determining but not determined by the foliation on the base space. This extra variable provides the flexibility needed for a functorial construction.

In fact, the characteristic class construction below compares the foliated bundle structure on a G-bundle P with another geometric structure on P, an H-reduction P' of P, where H is a closed subgroup of G. We wish to point out that this second piece of geometric structure is independent of the foliated bundle structure. In fact, the incompatibility of the two geometric structures is precisely what gives rise to these characteristic classes.

The starting point of this theory is the Chern-Weil theory in the form presented by Cartan in [CA]. A connection ω in a principal G-bundle $P \to M$ defines a Weil homomorphism

$$k(\omega)\colon W^{\cdot}(\mathfrak{g}) \to \Omega^{\cdot}(P) \tag{3.1}$$

into the De Rham complex $\Omega^{\cdot}(P)$ of global forms on P as follows. On the exterior part $\Lambda\mathfrak{g}^*$ in $W(\mathfrak{g}) \equiv \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$ the map $k(\omega)$ is simply the multiplicative extension of the (dual of the) connection form $\omega\colon \mathfrak{g}^* \to \Omega^1(P)$, assigning to $\alpha \in \mathfrak{g}^*$ the 1-form $\alpha\omega$. On the symmetric part $S\mathfrak{g}^*$ the map $k(\omega)$ is the multiplicative extension of the curvature

$$\Omega(\omega) = d_P\omega - \omega d_\Lambda\colon \mathfrak{g}^* \to \Omega^2(P).$$

The Weil algebra is equipped with a differential d_w (see e.g. [**KT 9**, pp. 55–57]) and $k(\omega)$ is a homomorphism of DG-algebras (differential graded algebras). This is not directly of interest in the classical theory, since $W(\mathfrak{g})$ is acyclic. The restriction $h(\omega) = k(\omega)|I(G)$ to the Ad-invariant polynomials

$$I(G) = (S\mathfrak{g}^*)^G$$

maps into the De Rham complex of M, viewed in canonical fashion as subcomplex of $\Omega^\cdot(P)$. Thus there is a commutative diagram

$$\begin{array}{ccc} W^\cdot(\mathfrak{g}) & \xrightarrow{\quad k(\omega) \quad} & \Omega^\cdot(P) \\ \cup & & \cup \\ I^\cdot(G) & \xrightarrow{\quad h(\omega) \quad} & \Omega^\cdot(M) \end{array} \tag{3.2}$$

The bottom map $h(\omega)$ is the Chern map, assigning to an Ad(G)-invariant polynomial Φ of degree p on $\mathfrak{g}^*$ the $2p$-form

$$h(\omega)\Phi = \Phi\Big(\underbrace{\Omega(\omega) \wedge \cdots \wedge \Omega(\omega)}_{p\text{-factors}}\Big) \in \Omega^{2p}(M)$$

where $\Omega \wedge \cdots \wedge \Omega$ is the $S^p\mathfrak{g}$-valued pth exterior product with itself of the $S^1\mathfrak{g}$-valued 2-form $\Omega = \Omega(\omega)$. Since $d_w\Phi = 0$, the form $h(\omega)\phi$ is closed. The De Rham cohomology class $[h(\omega)\phi] \in H^{2p}_{\mathrm{DR}}(M)$ is the characteristic class associated to Φ.

If ω is now an adapted connection on a foliated G-bundle, the Weil homomorphism $k(\omega)$ vanishes on the differential ideal $S^{q+1}\mathfrak{g}^* \cdot W(\mathfrak{g})$ of $W(\mathfrak{g})$ and thus passes to the quotient

$$W(\mathfrak{g})_q = W(\mathfrak{g})/S^{q+1}\mathfrak{g}^* \cdot W(\mathfrak{g}). \tag{3.3}$$

The integer q is the codimension of the given foliation on the base space. If ω is a basic connection, q can be replaced by the integer $[q/2]$. The truncated Weil algebra $W(\mathfrak{g})_q$ (resp. $W(\mathfrak{g})_{[q/2]}$) is not acyclic and thus its cohomology $H(W(\mathfrak{g})_q)$ gives rise to new characteristic invariants. So far only the foliated G-bundle structure intervened. If $H \subset G$ is further a closed subgroup, the relative version of this map is a DG-homomorphism

$$k(\omega)_H\colon W^\cdot(\mathfrak{g}, H)_q \to \Omega^\cdot(P/H) \tag{3.4}$$

landing in the De Rham complex of the quotient P/H (see [**KT 9**, Chapter 4] for the definition of the relative truncated Weil algebra $W(\mathfrak{g}, H)_q$ and more details of this construction). An H-reduction P' of P is characterized by a section $s\colon M \to P/H$ of the projection $P/H \to M$ in the form $P' \cong s^*P$ as H-bundles. The composition

$$\Delta(\omega) = s^* \circ k(\omega)_H\colon W^\cdot(\mathfrak{g}, H)_q \to \Omega^\cdot(M) \tag{3.5}$$

is a DG-homomorphism, which on the cohomology level defines the characteristic homomorphism $\Delta(P)_*$ of the foliated G-bundle with its H-reduction. This construction has the following properties [**KT 6**], [**KT 7**], [**KT 9**].

(3.6) THEOREM. *Let $\pi\colon P \to M$ be a foliated principal G-bundle, $H \subset G$ a closed subgroup and P' an H-reduction of P given by a section $s\colon M \to P/H$ of the induced map $\hat{\pi}\colon P/H \to M$.*

(i) *There is a well-defined multiplicative homomorphism*

$$\Delta(P)_*\colon H^{\cdot}(W(\mathfrak{g}, H)_q) \to H^{\cdot}{}_{\mathrm{DR}}(M)$$

where q is the codimension of the foliation on M. $\Delta_* = \Delta(P)_*$ is the generalized characteristic homomorphism of P.

(ii) *Δ_* does not depend on the choice of an adapted connection in P. But if P admits a basic connection, then*

$$\Delta(P)_*\colon H^{\cdot}(W(\mathfrak{g}, H)_{[q/2]}) \to H^{\cdot}{}_{\mathrm{DR}}(M).$$

(iii) *Δ_* is functorial under pullbacks and functorial in (G, H).*

(iv) *Δ_* is invariant under integrable homotopies.*

This construction applies in particular to G-foliations of codimension q, where the bundle P of G-frames is in addition equipped with an H-reduction. For ordinary $GL(q)$-foliation this additional geometric structure is often tacitly assumed to be an auxiliary metric or volume form on Q, i.e. and $O(q)$- or $SL(q)$-reduction. But many other cases of interest occur (see §6).

It is explained in **[KT 9**, pp. 71–72] how this construction relates to the construction of characteristic classes for foliations by Godbillon-Vey **[GV]**, Bott-Haefliger **[BH]**, **[H2]** and Bernstein-Rosenfeld **[BR1]**, **[BR2]** on the Gelfand-Fuchs complex of formal vectorfields (see also the comments towards the end of §7).

We observe that the ordinary Chern-Weil construction corresponds to the situation in Theorem 3.6 when $L = 0$ and $H = G$. Since $W(\mathfrak{g}, G) \cong I(G)$, the construction of $\Delta(P)_*$ reduces to the original definition of Chern.

The functoriality of Δ_* in (G, H) implies the following. If the foliation of the G-bundle P is induced by a foliation of the H-reduction P', then for an adapted connection ω in P' and its extension ω to P there is a commutative diagram

$$\begin{tikzcd} W(\mathfrak{g}, H)_q \arrow[d] \arrow[dr, "\Delta(\omega)"] & \\ I(H)_q \arrow[r, "\Delta(\omega')"'] & \Omega(P/H) \end{tikzcd}$$

Thus all one possibly recaptures in Δ_* is the ordinary Chern-Weil homomorphism of P'. The existence of further nontrivial classes in im Δ_* is thus a measure of the incompatibility of the two given geometric structures.

To make this precise, we need besides $W(\mathfrak{g}, H)_q$ the purely Lie algebraic object $W(\mathfrak{g}, \mathfrak{h})_q$ associated to the pair $(\mathfrak{g}, \mathfrak{h})$, namely the $\mathfrak{h}$-basic elements in $W(\mathfrak{g})_q$ (see **[KT 9]** for more details on these concepts). To compare the two algebras for a reductive pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras, we assume $H \subset G$ to be a closed subgroup with finitely many connected components. Then the component group Γ of H acts on $W(\mathfrak{g}, \mathfrak{h})_q$ and commutes with the Weil differential. Thus there is an induced action in the cohomology, and with respect to this action we have

$$H(W(\mathfrak{g}, H)_q) \cong \left(H(W(\mathfrak{g}, \mathfrak{h})_q)\right)^{\Gamma} \tag{3.7}$$

where the superscript Γ denotes the Γ-invariant elements in the cohomology $H(W(\mathfrak{g}, \mathfrak{h})_q)$. The following statement explains then the obstruction character of the characteristic classes **[KT 6]**, **[KT 7]**, **[KT 9]**.

(3.8) THEOREM. *Let P be a foliated G-bundle, $H \subset G$ a closed subgroup with finite component group Γ and P' an H-reduction of P. Assume $(\mathfrak{g}, \mathfrak{h})$ to be a reductive pair of Lie algebras.*

(i) *There is a split exact sequence of algebras*

$$0 \to \coprod_{-s>0} \operatorname{Tor}^{s,\cdot}_{I(\mathfrak{g})}(I(\mathfrak{h}), I(\mathfrak{g})_q) \to H(W(\mathfrak{g}, \mathfrak{h})_q) \underset{\kappa}{\leftrightarrows} I(\mathfrak{h}) \underset{I(\mathfrak{g})}{\otimes} I(\mathfrak{g})_q \to 0$$

and the composition $\Delta(P)_* \circ \kappa$ is induced by the characteristic homomorphism

$$h_*(P'): I(H) \to H_{\mathrm{DR}}(M) \text{ of } P'.$$

(ii) *If the foliation of the G-bundle P is induced by a foliation of the H-reduction P', then*

$$\Delta(P)_*| \coprod_{-s>0} \operatorname{Tor}^{s,\cdot}_{I(\mathfrak{g})}(I(\mathfrak{h}), I(\mathfrak{g})_q) = 0.$$

$\coprod_{-s>0} \operatorname{Tor}^{s,\cdot}_{I(\mathfrak{g})}(I(\mathfrak{h}), I(\mathfrak{g})_q)$ *is the algebra of secondary characteristic classes.*

This formulation anticipates the structure theorems for $H(W(\mathfrak{g}, \mathfrak{h})_q)$ given in §5. However before turning to this purely algebraic question, we consider in the next section the case $q = 0$ in more detail.

4. Flat bundles. A flat G-bundle $P \to M$ is a foliated bundle with $L = T_M$ on the base space M. The codimension q equals zero. For any subgroup $H \subset G$

$$W(\mathfrak{g}, H)_0 \cong \Lambda(\mathfrak{g}/\mathfrak{h})^{*H} \tag{4.1}$$

is the relative Chevalley-Eilenberg complex of (G, H) with the cohomology $H(\mathfrak{g}, H)$. The superscript H denotes H-invariant elements under the adjoint action. Thus for any closed subgroup $H \subset G$ and H-reduction $s: M \to P/H$ the characteristic homomorphism of the flat bundle is a map

$$\Delta(P)_*: H(\mathfrak{g}, H) \to H_{\mathrm{DR}}(M). \tag{4.2}$$

We assume that the group Γ of components of H is finite. Then we have as in (3.7) the formula

$$H(\mathfrak{g}, H) \cong H(\mathfrak{g}, \mathfrak{h})^{\Gamma}. \tag{4.3}$$

The RHS denotes the Γ-invariant elements in the relative Chevalley-Eilenberg cohomology $H(\mathfrak{g}, \mathfrak{h})$ of the pair $(\mathfrak{g}, \mathfrak{h})$. This is the cohomology of the complex $\Lambda(\mathfrak{g}/\mathfrak{h})^{*\mathfrak{h}}$ ($\mathfrak{h}$-invariant elements), on which Γ obviously acts in differential fashion, and thus induces an action in $H(\mathfrak{g}, \mathfrak{h})$.

The determination of $H(\mathfrak{g}, \mathfrak{h})$ for a reductive pair $(\mathfrak{g}, \mathfrak{h})$ involves the restriction homomorphism $i^*\colon I(\mathfrak{g}) \to I(\mathfrak{g})$ and the Samelson space $\hat{P} = \hat{P}(\mathfrak{g}, \mathfrak{h})$ of $(\mathfrak{g}, \mathfrak{h})$. To recall the definition of the latter space, let $P_\mathfrak{g}$ denote the primitive elements of $H(\mathfrak{g})$. Consider the inclusion $j\colon \Lambda(\mathfrak{g}/\mathfrak{h})^{*\mathfrak{h}} \subset \Lambda\mathfrak{g}^*$ and the induced cohomology map $j_*\colon H(\mathfrak{g}, \mathfrak{h}) \to H(\mathfrak{g})$. Then

$$\hat{P} = P_\mathfrak{g} \cap \operatorname{im} j_*. \tag{4.4}$$

Cartan pairs (C-pairs) $(\mathfrak{g}, \mathfrak{h})$ are characterized by the identity

$$\dim \hat{P} = \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{h}. \tag{4.5}$$

For C-pairs there is a well-known isomorphism (see **[CA]**, **[KT 9]** and (5.18) below)

$$H(\mathfrak{g}, \mathfrak{h}) \cong \Lambda\hat{P} \otimes I(\mathfrak{h})/\mathrm{Id}\big(i^*I(\mathfrak{g})^+\big). \tag{4.6}$$

The factor $I(\mathfrak{h})/\mathrm{Id}(i^*I(\mathfrak{g})^+)$ is the image of the canonical homomorphism $I(\mathfrak{h}) \to H(\mathfrak{g}, \mathfrak{h})$. The map $\Delta(P)_*$ restricted to this factor in $H(\mathfrak{g}, H)$ is determined by the usual characteristic homomorphism $h_*\colon I(H) \to H_{\mathrm{DR}}(M)$ of the H-reduction P'. In case of a surjective restriction $i^*\colon I(\mathfrak{g}) \to I(\mathfrak{h})$ the invariants arising from $\Lambda\hat{P}$ are the only invariants occurring. This is the situation in several examples below.

4.7. $(G, H) = (GL(m), O(m))$. The restriction homomorphism

$$i^*\colon \mathbf{R}[c_1, \dots, c_m] \cong I(\mathfrak{gl}(m)) \to \mathbf{R}[p_1, \dots, p_{[m/2]}] \subset I(\mathfrak{so}(m))$$

is characterized by

$$i^*c_{2j-1} = 0, \quad i^*c_{2j} = p_j \quad \text{for } j = 1, \dots, [m/2],$$

where c_k and p_l denote the Chern and Pontrjagin polynomials respectively. This pair is an example of a Special Cartan pair (see Lemma 5.9 below), for which the Samelson space $\hat{P}$ is the image of $\ker i^*$ under the suspension map $\sigma\colon I(\mathfrak{g}) \to P_\mathfrak{g}$. It follows that with the primitive generators $y_j = \sigma c_j \in H(\mathfrak{gl}(m))$ the space $\hat{P}$ is spanned by $y_1, y_3, \dots, y_{m'}$ (m' is the largest odd number $\leqslant m$). The group $Z_2 = \pi_0(O(m))$ clearly does not act on $\Lambda\hat{P}$. On $I(\mathfrak{so}(m))$ it only acts nontrivially on the Pfaffian polynomial e_m with $p_{[m/2]} = e_m^2$ in case $m = 2n$. By (4.3) and (4.6) it follows that for any m

$$H(\mathfrak{gl}(m), O(m)) \cong \Lambda(y_1, y_3, \dots, y_{m'}). \tag{4.8}$$

Since $GL(m)/O(m)$ is contractible, an $O(m)$-reduction exists and its particular choice does not affect $\Delta(P)_*$. The map (4.2) leads then to the following result (see **[KT 7]** and **[KT 9**, 6.33**]**).

4.9 Theorem. *Let $P \to M$ be a flat $GL(m)$-bundle. There are well-defined characteristic classes*

$$\Delta(P)_*(y_i) \in H_{\mathrm{DR}}^{2i-1}(M) \quad \textit{for } i = 1, 3, \dots, m', m' = 2[m + 1/2] - 1.$$

If P is $O(m)$-flat, all these classes are trivial.

Since the flat bundle P is completely characterized by the holonomy representation $h\colon \pi_1(M) \to GL(m)$, it would be interesting to determine the invariants $\Delta_*(y_i)$ from h. For the invariant $\Delta_*(y_1) \in H_{\mathrm{DR}}^1(M)$ this is done by the following formula (see **[KT 7]** and **[KT 9**, 6.34**]**).

4.10 PROPOSITION. *Let P be a flat $GL(m)$-bundle with connection form ω. Then $\Delta_*(y_1)$ is represented by a closed 1-form $\Delta(\omega)(y_1)$ on M and*

$$\int_\gamma \Delta(\omega)(y_1) = -\frac{1}{2\pi}\log|\det h(\gamma)| \quad \text{for } \gamma \in \pi_1(M).$$

This formula shows that $\Delta_*(y_1)$ is nonzero if and only if the holonomy representation does not map into the $(m \times m)$-matrices with determinant ± 1. In the following situation one obtains a nontrivial realization of the invariant $\Delta_*(y_1)$.

4.11 PROPOSITION [KT 10], [KT 9, 6.39]. *Let M^m be a compact affine hyperbolic manifold. Then for the tangentbundle $\Delta_*(y_1)$ is a nontrivial cohomology class.*

The hyperbolicity of the affine structure means that the universal covering of M^m is affinely isomorphic to an open convex subset of $\mathbf{R}^m$ containing no complete line (these are noncomplete affine manifolds, see Koszul [K 3]).

What the formula in (4.10) does is to represent $\Delta_*(y_1)$ by a cocycle in the Eilenberg-Mac Lane cohomology of $\Gamma = \pi_1(M)$. The question arises how to represent more generally characteristic classes of flat bundles as in (2.27) with $X = B\Gamma$ (classifying space of a discrete group Γ) as cocycles in the Eilenberg-Mac Lane cohomology $H(\Gamma, \mathbf{R}) \cong H_{\mathrm{DR}}(B\Gamma)$. Recently some formulas of this type have been found by Dupont [DU] and Bott [B 9].

4.12 $(G, H) = (GL(m), SL(m))$. The restriction homomorphism

$$i^*: I(\mathfrak{gl}(m)) \cong \mathbf{R}[c_1, \dots, c_m] \to I(\mathfrak{sl}(m)) \cong \mathbf{R}[c_2, \dots, c_m]$$

is characterized by $i^*c_1 = 0$ and $\hat{P}$ is spanned by $y_1 = \sigma c_1 \in H(\mathfrak{gl}(m))$. By (4.3) and (4.6) therefore

$$H(\mathfrak{gl}(m)\, SL(m)) \cong \Lambda(y_1).$$

The same formula as above applies for the characteristic class $\Delta(P)_*(y_1)$ of a flat $GL(m)$-bundle with an $SL(m)$-reduction.

4.13 $(G, H) = (GL(m, \mathbf{C}), SL(2, \mathbf{C}))$. Let i: $SL(2, \mathbf{C}) \to GL(m, \mathbf{C})$ be an injective homomorphism, i.e. a nontrivial representation of $SL(2, \mathbf{C})$ in $\mathbf{C}^m$. Consider the restriction homomorphism

$$i^*: I(\mathfrak{gl}(m, \mathbf{C})) \cong \mathbf{C}[c_1, \dots, c_m] \to I(\mathfrak{sl}(2, \mathbf{C})) \cong \mathbf{C}[\bar{c}_2].$$

Since $I(\mathfrak{sl}(2, \mathbf{C}))/I(\mathfrak{gl}(m, \mathbf{C}))^+ \cdot I(\mathfrak{sl}(2, \mathbf{C}))$ is finite dimensional, some nonzero power of $\bar{c}_2$ is hit by i^*. Let j be the smallest such exponent, $j > 1$. Then clearly

$$i^*c_{2j} = \lambda \bar{c}_2^j \quad \text{for some } \lambda \neq 0.$$

Furthermore $i^*c_{2k-1} = 0$ for $k = 1, \dots, [m/2]$ and for $k > j$

$$i^*c_{2k} = \lambda_k \bar{c}_2^k = (\lambda_k/\lambda) \cdot i^*c_{2j} \cdot \bar{c}_2^{k-j}.$$

By (5.5) below it follows that $\sigma c_{2k} \in \hat{P}$ for all $k \neq j$ and hence the Samelson space of the pair $(\mathfrak{gl}(m, \mathbf{C}), i\mathfrak{sl}(2, \mathbf{C}))$ is given by

$$\hat{P} = \{y_1, \dots, \hat{y}_{2j}, \dots, y_m\}$$

where y_l is the primitive class corresponding to c_l. Since

$$\dim \hat{P} = \operatorname{rank} \mathfrak{gl}(m, \mathbf{C}) - \operatorname{rank} \mathfrak{sl}(2, \mathbf{C}) = m - 1$$

it follows that $(\mathfrak{gl}(m, \mathbf{C}), i\mathfrak{sl}(2, \mathbf{C}))$ is a Cartan pair (see (4.5)). It is worth noting that if $j > 1$ and there exists a $\lambda_k \neq 0$ as above for some k with $k < 2j$, then certainly $\bar{c}_2^{k-j} \notin \operatorname{im} i^*$. Such a pair is then an example of a C-pair which is not a CS-pair (the latter concept is discussed in §5, see Lemma 5.9).

From (4.6) we conclude now that

$$H(\mathfrak{gl}(m, \mathbf{C}), i\mathfrak{sl}(2, \mathbf{C})) \cong \Lambda(y_1, \dots, \hat{y}_{2j}, \dots, y_m) \otimes \mathbf{C}[\bar{c}_2]/(\bar{c}_2^j). \quad (4.14)$$

For a flat complex vectorbundle $E \to M$ of fiber-dimension m with an $SL(2, \mathbf{C})$-reduction there are therefore characteristic classes $\Delta(E)_*(y_i) \in H_{\mathrm{DR}}^{2i-1}(M)$ for $i \neq 2j$. The class $\Delta(E)_*(\bar{c}_2)$ is the second Chern class of the $SL(2, \mathbf{C})$-reduction (its unique nontrivial Chern class), and hence

$$\bar{c}_2(E)^j = 0.$$

Such a situation arises in the theory of deformations of Kähler manifolds. The vectorbundle in question is the cohomology vectorbundle with its flat Gauss-Manin connection.

4.15 $(G, H) = (U(m), O(m))$. With the modified Chern polynomial $\tilde{c}_j = i^j c_j$ we have $I(\mathfrak{u}(m)) \cong \mathbf{R}[\tilde{c}_1, \dots, \tilde{c}_m]$ (see [**KT 9**, p. 138] for details on these normalizations). The restriction map

$$i^*: I(\mathfrak{u}(m)) \cong \mathbf{R}[\tilde{c}_1, \dots, \tilde{c}_m] \to \mathbf{R}[p_1, \dots, p_{[m/2]}] \subset I(\mathfrak{so}(m))$$

is characterized by

$$i^*\tilde{c}_{2j-1} = 0, \; i^*\tilde{c}_{2j} = (-1)^j p_j \quad \text{for } j = 1, \dots, [m/2].$$

The Samelson space $\hat{P}$ is spanned by $y_1, y_3, \dots, y_{m'}$ as before, where now $y_j = \sigma\tilde{c}_j$. Thus by (4.3), (4.6)

$$H(\mathfrak{u}(m), O(m)) \cong \Lambda(y_1, y_3, \dots, y_{m'}).$$

The map (4.2) leads then to the following result (see [**KT 9**]–[**KT 11**]).

4.16 THEOREM. *Let $P' \to M$ be an $O(m)$-bundle with a flat $U(m)$-extension. There are well-defined characteristic classes*

$$\Delta(P)_*(y_i) \in H_{\mathrm{DR}}^{2i-1}(M) \quad \textit{for } i = 1, 3, \dots, m', \; m' = 2[(m + 1)/2] - 1.$$

This applies e.g. for real vector bundles with a flat complexification. If the $U(m)$-extension is in particular trivial, the characteristic homomorphism $\Delta(P)_*$ has the following geometric interpretation. Let g': $M \to BO(m)$ be the classifying map of P. The classifying map g: $M \to BU(m)$ of the $U(m)$-extension is homotopic to a constant map. Since g is the composition of g' with $BO(m) \to BU(m)$, it follows that g' factorizes through the fiber $U(m)/O(m)$

$$\begin{tikzcd} U(m)/O(m) \arrow[r] & BO(m) \arrow[r] & BU(m) \\ & M \arrow[ul, "f"] \arrow[u, "g'"] \arrow[ur, "g"'] & \end{tikzcd}$$

Then $\Delta(P)_*$ is the composition

$$H(\mathfrak{u}(m), O(m)) \xrightarrow{\cong} H_{DR}(U(m)/O(m)) \xrightarrow{f^*} H_{DR}(M)$$

where the first isomorphism is realized by the inclusion of the $O(m)$-invariant forms into $\Omega(U(m)/O(m))$. The class $\Delta(P)_*(y_1)$ of Theorem 4.16 is in this case (up to a factor) the Maslov class which enters in quantization conditions (see [A] or the appendix to [MS]).

4.17 PROPOSITION [KT 9]-[KT 11]. *Let $P' \to M$ be an $O(m)$-bundle with a trivial $U(m)$-extension P, characterized by f: $M \to U(m)/O(m)$. Then the Maslov class of P' is the characteristic class $-2\Delta_*(y_1) \in H^1_{DR}(M)$. If ω is the connection form of the trivial connection on P, then $\Delta_*(y_1)$ on M and*

$$-2\int_\gamma \Delta(\omega)(y_1) = \deg\left(\det{}^2 \circ f(\gamma)\right) \quad \text{for } \gamma \in \pi_1(M).$$

Here γ: $S^1 \to M$ and $f(\gamma)$: $S^1 \to U(m)/O(m)$. The map $\det^2$ is the map det: $U(m) \to S^1$ squared, which factorizes through $U(m)/O(m)$. Thus $\det^2 \circ f(\gamma)$: $S^1 \to S^1$ is a mapping of the circle, of which the RHS takes the degree.

4.18 $(G, H) = (SO(2m), SO(2m-1))$. The restriction homomorphism

$$i^*: I(\mathfrak{so}(em)) \cong \mathbf{R}[p_1, \ldots, p_{m-1}, e_m] \to I(\mathfrak{so}(2m-1)) \cong \mathbf{R}[p_1, \ldots, p_{m-1}]$$

is characterized by $i^* e_m = 0$ for the Pfaffian polynomial e_m ($e_m^2 = p_m$), and thus

$$H(\mathfrak{so}(2m), SO(2m-1)) \cong \Lambda(\sigma e_m),$$

where σe_m is the suspension of e_m. It follows that the only interesting class of a flat $SO(2m)$-bundle with $SO(2m-1)$-reduction is $\Delta_*(\sigma e_m)$. The following situation illustrates this (see [KT 9]-[KT 11]).

4.19 THEOREM. *Let h: $M^{2m-1} \to \mathbf{R}^{2m}$ be an isometric immersion of the compact oriented Riemannian manifold M. The $SO(2m-1)$-frame bundle P' of M has a trivial $SO(2m)$-extension. Then the characteristic class $\Delta_*(\sigma e_m) \in H^{2m-1}_{DR}(M)$ can be evaluated on M and*

$$N(h) = (-1)^m \cdot 2^{2(m-1)} \langle \Delta_*(\sigma e_m), [M] \rangle \tag{4.20}$$

where $N(h)$ is the normal degree of h.

The numerical factor on the RHS of (4.20) was misstated in [KT 9]-[KT 11]. This formula is to be contrasted with Hopf's formula for the normal degree of an immersion h: $M^{2m} \to \mathbf{R}^{2m+1}$, which states that $N(h) = \frac{1}{2}\chi(M)$, where $\chi(M)$ is the Euler number of M. Hopf's formula is of a primary nature and valid for any immersion, whereas (4.20) is of secondary nature and valid only for isometric immersions. Theorem 4.19 is of course very much in the spirit of the invariants defined by Chern-Simons [CS].

4.21. A situation of considerable interest is the following. Let $K \subset G$ be a maximal compact subgroup of a Lie group G and consider the flat G-bundle

$$G \times_K G \cong G/K \times G \to G/K. \tag{4.22}$$

The flat structure is induced by the diagonal action of G (see 2.20). This bundle is obviously the canonical G-extension of the K-bundle $G \to G/K$, which hence is a K-reduction of the flat G-bundle. Let $\Gamma \subset G$ be a discrete uniform subgroup operating properly discontinuously and without fixed points on G/K, so that the double coset space $\Gamma \setminus G/K$ is a manifold. By Borel [BO] such a Γ exists if G is connected semisimple with finite center and no compact factor. The flat G-bundle

$$P = (\Gamma \setminus G) \times_K G \cong G/K \times_\Gamma G \to M = \Gamma \setminus G/K \tag{4.23}$$

on the Clifford-Klein form $\Gamma \setminus G/K$ of the noncompact symmetric space G/K is then canonically equipped with a reduction to the K-bundle $\Gamma \setminus G \to \Gamma \setminus G/K$. The characteristic homomorphism

$$\Delta(P)_*\colon H(\mathfrak{g}, K) \to H_{\mathrm{DR}}(M) \tag{4.24}$$

is then well defined. Since G/K is contractible, it follows that the Clifford-Klein form M is a classifying space $B\Gamma$ for the discrete group Γ: $M \simeq B\Gamma$. Therefore $H(M) \cong H(B\Gamma)$, which is the cohomology $H(\Gamma)$ of the discrete group Γ. The inclusion $\alpha\colon \Gamma \subset G$ induces a map $B\alpha\colon B\Gamma \to BG$ of classifying spaces. The K-bundle $P'\colon \Gamma \setminus G \to \Gamma \setminus G/K$ is classified by a map $g'\colon M \to BK$. Since for compact K the universal characteristic map $h_*\colon I(K) \to H(BK)$ is an isomorphism, the map $g'^*\colon H(BK) \to H(M)$ can be identified with the characteristic homomorphism of P'. Let similarly $h_*(G, K)\colon H(BK) \to H(\mathfrak{g}, K)$ be identified with the characteristic homomorphism of $G \to G/K$. Then there is the following commutative diagram ([KT 1, (4.18)])

$$\begin{tikzcd}
& H(BG) \arrow[r, "\cong"] \arrow[d, "B\alpha^*"'] & H(BK) \arrow[d, "h_*(G,K)"] \arrow[dl, "g'^*"'] \\
H(\Gamma) \cong & H(M) \arrow[r, "\Delta(P)_*"'] & H(\mathfrak{g}, K)
\end{tikzcd} \tag{4.25}$$

The isomorphism $H(BG) \xrightarrow{\cong} H(B_k)$ is a consequence of the homotopy equivalence $K \simeq G$. Note that the map $\Delta(P)_*$ is induced by the canonical inclusion

$$(\Lambda\mathfrak{g}^*)_K \to \Omega^\cdot(\Gamma \setminus G/K)$$

so it really is a tautological map. Diagram 4.25 relates the existence of nontrivial classes under $\Delta(P)_*$ with the existence of nontrivial classes under the map $B\alpha^*$. The following result is in particular useful for the detection of nontrivial characteristic classes of quotient foliations as described in 2.23 (see e.g. [KT 6], [KT 7], [KT 11]). Its proof is based on Lemma 4.21 of [KT 1].

4.26 THEOREM. *Let G be a connected semisimple Lie group with finite center and containing no compact factor, $K \subset G$ a maximal compact subgroup and*

$\Gamma \subset G$ a discrete, uniform and torsion-free subgroup. Then the generalized characteristic homomorphism

$$\Delta_*: H(\mathfrak{g}, K) \to H_{\mathrm{DR}}(M)$$

of the flat bundle

$$P = \Gamma \setminus G \times_K G \cong G/K \times_\Gamma G \to \Gamma \setminus G/K$$

is injective.

We proceed to give a more geometric interpretation of these classes. Let $G_{\mathbf{C}}$ be the complexification of G and $U \subset G_{\mathbf{C}}$ a maximal compact subgroup. Then

$$H(\mathfrak{g}, K) \cong H(\mathfrak{u}, K) \cong H(U, K)$$

so that the elements of $H(\mathfrak{g}, K)$ can be realized by cohomology classes of the compact space U/K (whereas G/K is contractible). A typical example is $G = SL(n, \mathbf{R})$ with complexification $SL(n, \mathbf{C})$. In this case $K = SO(n)$ and $U = SU(n)$. The map $\Delta(P)_*$ is then realized on the cochain level by the map

$$\gamma: (\Lambda\mathfrak{u}^*)_K \to \Omega(\Gamma \setminus G/K)$$

which is exactly Matsushima's map constructed in **[MT]**: an invariant form on U/K is characterized by an element in $(\Lambda\mathfrak{u}^*)_K$, which canonically defines an element in $(\Lambda\mathfrak{g}^*)_K$, which in turn defines a G-invariant form on G/K, hence a form in $\Omega(\Gamma \setminus G/K)$. Since both the form we start with and the form we end up with are harmonic, this map realizes the induced map on the cohomology level, and the injectivity in cohomology is obvious.

For an interpretation of the map $\Delta(P)_*$ as a generalized proportionality map between the characteristic classes of the vectorbundles on $\Gamma \setminus G/K$ and U/K associated to a K-module we refer to p. 22 of **[KT 1]** and p. 92 of **[KT 9]**.

5. Universal characteristic classes. The algebra $H(W(\mathfrak{g}, H)_q)$ plays the role of an algebra of universal characteristic classes for foliated G-bundles $P \to M$ with q the codimension of $L \subset T_M$ and equipped with an H-reduction. Its role is analogous to the role of the cohomology algebra $H(BG)$ of the classifying space BG for ordinary G-bundles. In case a basic connection exists, the relevant algebra is $H(W(\mathfrak{g}, H)_{[q/2]})$.

The relation between the algebra $H(W(\mathfrak{g}, H)_k)$ associated to the pair (G, H) of groups and the algebra $H(W(\mathfrak{g}, \mathfrak{h})_k)$ associated to the pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras is explained in (3.7). In this section we discuss the purely algebraic problem of the computation of $H(W(\mathfrak{g}, \mathfrak{h})_k)$ for reductive pairs $(\mathfrak{g}, \mathfrak{h})$ and any integer $k \geqslant 0$. The algorithm presented here is based on **[KT 5]**, **[KT 7]**, **[KT 14]**, **[KT 15]**. A detailed account in the natural context of $\mathfrak{g}$-DG-algebras can be found in **[KT 16]**. Applications of these results to G-foliations are then discussed in §§6 and 7.

Let $(\mathfrak{g}, \mathfrak{h})$ be a reductive pair of Lie algebras over the groundfield K of characteristic zero, with inclusion map $i: \mathfrak{h} \subset \mathfrak{g}$. The suspension map $\sigma: I(\mathfrak{g})^+ \to H(\mathfrak{g})$ has as image the space of primitive elements $P_\mathfrak{g} \subset H(\mathfrak{g})$ and $\ker \sigma = (I(\mathfrak{g})^+)^2$. Let $\tau_\mathfrak{g}: P_\mathfrak{g} \to I(\mathfrak{g})$ denote any transgression ($\sigma \circ \tau_\mathfrak{g} = \mathrm{id}$). Then $V = \tau_\mathfrak{g} P_\mathfrak{g} \subset I(\mathfrak{g})$ represents the space of indecomposable elements and it is well known that $S(V) \cong I(\mathfrak{g})$, where $S(V)$ denotes the symmetric algebra over V. We have further for the relative cohomology of $(\mathfrak{g}, \mathfrak{h})$ the isomorphism [CA]

$$H(\mathfrak{g}, \mathfrak{h}) \cong H(\Lambda P \otimes I(\mathfrak{h})). \tag{5.1}$$

The differential on the RHS is characterized by $d = 0$ on $I(\mathfrak{h})$ and

$$d(y \otimes 1) = -1 \otimes i^*\tau_{\mathfrak{g}} y \quad \text{for } y \in P = P_{\mathfrak{g}}.$$

Consider the commutative diagram

$$\begin{tikzcd} I(\mathfrak{g}) \arrow[r, "i^*"] & I(\mathfrak{h}) \arrow[r, "h_*"] \arrow[dr, "(\mathrm{incl})_*"'] & H(\mathfrak{g}, \mathfrak{h}) \arrow[r, "j_*"] & H(\mathfrak{g}) \\ & & H(\Lambda P \otimes I(\mathfrak{h})) \arrow[u, "\cong"] \arrow[ur, "(\mathrm{proj})_*"'] & \end{tikzcd} \tag{5.2}$$

This sequence is exact at $I(\mathfrak{h})$ in the sense that

$$\ker h_* = \mathrm{Id}\big(i^*I(\mathfrak{g})^+\big). \tag{5.3}$$

Furthermore

$$\ker j_* \supset h_* I(\mathfrak{h})^+. \tag{5.4}$$

Using (5.2), the Samelson space $\hat{P}$ as defined in (4.4) can also be characterized by

$$\hat{P} = \left\{ y \in P_{\mathfrak{g}} \middle| i^*\tau_{\mathfrak{g}} y = \sum_k i^*\tau_{\mathfrak{g}} y_k \cdot \phi_k, y_k \in P, \phi_k \in I(\mathfrak{h})^+ \right\}. \tag{5.5}$$

We will need the following condition on the pair $(\mathfrak{g}, \mathfrak{h})$:

$$\hat{P} \subset \sigma \ker(i^*: I(\mathfrak{g}) \to I(\mathfrak{h})). \tag{5.6}$$

By (5.5) the reverse inclusion always holds and therefore (5.6) is equivalent to $\hat{P} = \sigma(\ker i^*)$. Also (5.6) is equivalent to the condition:

there exists a transgression $\tau_{\mathfrak{g}}$ such that

$$\tau_{\mathfrak{g}} \hat{P} \subset \ker i^*. \tag{5.7}$$

It follows from (5.3) that $I(\mathfrak{h})/\mathrm{Id}(i^*I(\mathfrak{g})^+) \subset H(\mathfrak{g}, \mathfrak{h})$ and hence $I(\mathfrak{h})/\mathrm{Id}(i^*I(\mathfrak{g})^+)$ is finite dimensional. From this one concludes that the deficiency

$$d = \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{h} - \dim \hat{P} \tag{5.8}$$

of $(\mathfrak{g}, \mathfrak{h})$ satisfies $d \geqslant 0$. A Cartan pair (C-pair) $(\mathfrak{g}, \mathfrak{h})$ is by (4.5) a reductive pair satisfying $d = 0$. A slightly more restricted class of reductive pairs has been introduced in [KT 5], [KT 7], [KT 9]. A special Cartan pair (CS-pair) is a reductive pair satisfying one and hence both conditions in the following lemma.

5.9 LEMMA. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a reductive pair of Lie algebras. The following conditions are equivalent*:

$$\hat{P} = \sigma(\ker i^*) \quad \textit{and} \quad d = 0 \tag{5.10}$$

$$\mathrm{Id}\left(\tau_{\mathfrak{g}}\hat{P}\right) = \ker i^* \quad \textit{for some transgression } \tau_{\mathfrak{g}}. \tag{5.11}$$

Many of the familiar reductive pairs are CS-pairs, e.g. all the symmetric pairs. For an example of a C-pair which is not a CS-pair, see (4.13).

In the following we denote by $\hat{A}_k$, $k \geqslant 0$, the DG-algebra

$$\hat{A}_k = \Lambda\hat{P} \otimes I(\mathfrak{g})_k \tag{5.12}$$

with differential characterized by $d_{\hat{A}}(y \otimes 1) = 1 \otimes \tau_{\mathfrak{g}} y$ for $y \in \hat{P}$ and $d_{\hat{A}}(1 \otimes \phi) = 0$. If $\tilde{P}$ denotes a complement of $\hat{P}$ in P, we also have a decomposition $V = \hat{V} \oplus \tilde{V}$, $\tau_{\mathfrak{g}}\hat{P} = \hat{V}$, $\tau_{\mathfrak{g}}\tilde{P} = \tilde{V}$ of the space V of indecomposable elements. Since the canonical map $I(\mathfrak{g}) \cong S(\hat{V}) \otimes S(\tilde{V}) \to H(\hat{A}_k)$ is zero on $\mathrm{Id}(\hat{V})$, it follows that $H(\hat{A}_k)$ is canonically a module over $I(\mathfrak{g})/\mathrm{Id}(\hat{V}) \cong S(\tilde{V})$. The principal structure theorem for $H(W(\mathfrak{g}, \mathfrak{h})_k)$ can now be formulated.

5.13. THEOREM. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a reductive pair of Lie algebras over a groundfield K of characteristic zero, and* $k \geqslant 0$ *an integer.*

(i) *There is a canonical isomorphism of algebras*

$$H(W(\mathfrak{g}, \mathfrak{h})_k) \cong \mathrm{Tor}_{I(\mathfrak{g})}(I(\mathfrak{h}), I(\mathfrak{g})_k). \tag{5.14}$$

Moreover, $\mathrm{Tor}^{s,\cdot}_{I(\mathfrak{g})}(I(\mathfrak{h}), I(\mathfrak{g})_k) = 0$ *for* $-s > \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{h}$.

(ii) *There is a canonical isomorphism of algebras*

$$H(\hat{A}_k) \cong \mathrm{Tor}_{I(\mathfrak{g})}\left(S(\tilde{V}), I(\mathfrak{g})_k\right). \tag{5.15}$$

(iii) *Assume that if* $k > 0$, *the pair* $(\mathfrak{g}, \mathfrak{h})$ *satisfies* (5.6), *i.e.,* $\hat{P} \subset \sigma(\ker i^*)$ (*no assumption is necessary for* $k = 0$). *Then there exists a multiplicative spectral sequence*

$$E_2^{s,t} = \mathrm{Tor}^{s,t}_{S(\tilde{V})}\left(I(\mathfrak{h}), H(\hat{A}_k)\right) \Rightarrow H^{s+t}(W(\mathfrak{g}, \mathfrak{h})_k), \tag{5.16}$$

whose nonzero terms $E_2^{s,t}$ *are in the range* $0 \leqslant -s \leqslant d$.

5.17 REMARK. For the graded Tor-functor we use the same degree conventions as in [BA], i.e.

$$\mathrm{Tor} = \coprod_{s \leqslant 0,\, t \geqslant 0} \mathrm{Tor}^{s,t}$$

so that e.g. in (5.14) we have

$$H^n(W(\mathfrak{g}, \mathfrak{h})_k) \cong \coprod_{s+t=n} \mathrm{Tor}^{s,t}_{I(\mathfrak{g})}(I(\mathfrak{h}), I(\mathfrak{g})_k).$$

Theorem 5.13 still holds for $k = \infty$ with $W(\mathfrak{g}, \mathfrak{h})_\infty = W(\mathfrak{g}, \mathfrak{h})$. In particular $H(\hat{A}_\infty) \cong S(\tilde{V})$ and $H(W(\mathfrak{g}, \mathfrak{h})) \cong I(\mathfrak{h})$. For $k = 0$ we have $W(\mathfrak{g}, \mathfrak{h})_0 \cong \Lambda(\mathfrak{g}/\mathfrak{h})^{*\mathfrak{h}}$ and $\hat{A}_0 = \Lambda\hat{P}$. In this case we obtain the following consequence.

5.18 COROLLARY. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a reductive pair of Lie algebras. Then*

$$H(\mathfrak{g}, \mathfrak{h}) \cong \Lambda\hat{P} \otimes \left[I(\mathfrak{h})/\mathrm{Id}(i^* I(\mathfrak{g})^+) \oplus \coprod_{-s=1}^{d} \mathrm{Tor}^{s,\cdot}_{S(\tilde{V})}(I(\mathfrak{h}), K) \right]. \tag{5.19}$$

The deficiency d of (5.8) *is characterized as the largest integer for which* $\operatorname{Tor}^{d,\cdot}_{S(\hat{V})}(I(\mathfrak{h}), K) \neq 0$. *In particular* $(\mathfrak{g}, \mathfrak{h})$ *is a C-pair* ($d = 0$) *if and only if*

$$\operatorname{Tor}^{s,\cdot}_{S(\hat{V})}(I(\mathfrak{h}), K) = 0 \quad \textit{for } -s > 0.$$

This is a well-known result going back to Cartan-Koszul **[CA]**, **[K2]** and Baum **[BA]**.

5.20. COROLLARY. *If* $(\mathfrak{g}, \mathfrak{h})$ *is a CS-pair, then*

$$H(W(\mathfrak{g}, \mathfrak{h})_k) \cong H(\hat{A}_k) \otimes_{I(\mathfrak{g})} I(\mathfrak{h}). \tag{5.21}$$

In the remainder of this section we explain how part (ii) of Theorem 5.13 leads directly to an explicit computation of $H(\hat{A}_k)$.

Let $r = \operatorname{rank} \mathfrak{g}$ and choose generators c_i ($i = 1, \ldots, r$) so that $I(\mathfrak{g}) \cong K[c_1, \ldots, c_r]$. For the case $\mathfrak{g} = \mathfrak{gl}(r, K)$ the c_i's are the Chern polynomials. In the general case we assume that the polynomials c_i are ordered by increasing degree, i.e. $\deg c_i \leqslant \deg c_k$ for $i \leqslant k$.

Let $y_1, \ldots, y_{r'}$ be the basis of $\hat{P}$ such that y_l transgresses to c_{α_l} ($\alpha_1 \leqslant \cdots \leqslant \alpha_{r'}$). For $\mathfrak{h} = 0$ in particular $\hat{P} = P$, $r' = r$ and $\alpha_l = l$ for all l. With these notations we have

$$\hat{A}_k = \Lambda(y_1, \ldots, y_{r'}) \otimes K[c_1, \ldots, c_r]_k \tag{5.22}$$

with differential characterized by $dy_l = c_{\alpha_l}$, $dc_i = 0$. We use the following conventions:

$y_{(i)} = y_{i_1} \wedge \cdots \wedge y_{i_s}$ for $(i) = (i_1, \ldots, i_s)$, $1 \leqslant i_1 < \cdots < i_s \leqslant r'$ $(s > 0)$;

$y_{(i)} = 1$ for $(i) = \varnothing$ $(s = 0)$;

$c_{(j)} = c_1^{j_1} \cdots c_r^{j_r}$ for $(j) = (j_1, \ldots, j_r)$, $0 \leqslant j_i$;

$2p = \deg c_{(j)} = \sum_{i=1}^{r} j_i \deg c_i$.

Define the subspace

$$Z^{-s} \subset \Lambda_s(y_1, \ldots, y_{r'}) \otimes K[c_1, \ldots, c_r]_k \tag{5.23}$$

as the space generated by the monomials cochains

$$z_{(i,j)} = y_{(i)} \otimes c_{(j)} \tag{5.24}$$

satisfying the conditions

$$0 \leqslant 2p \leqslant 2k, \qquad 0 \leqslant s \leqslant r'; \tag{5.25}$$

$$\deg c_{\alpha_{i_1}} = \deg y_{i_1} + 1 \geqslant 2(k + 1 - p) \quad \text{if } (i) \neq \varnothing \ (s > 0); \tag{5.26}$$

$$\begin{aligned} j_{\alpha_l} &= 0 \quad \text{for } l < i_1, \text{ if } (i) \neq \varnothing \text{ and} \\ j_{\alpha_l} &= 0 \quad \text{for all } l, \text{ if } (i) = \varnothing. \end{aligned} \tag{5.27}$$

It is clear that the $z_{(i,j)}$'s are cocycles and therefore $dZ^{-s} = 0$. The sum $Z_k = \coprod_{s=0}^{r'} Z^{-s}$ is a subalgebra of $\hat{A}_k$. The products of monomial cocycles $z_{(i,j)}$ satisfying $j_{\alpha_l} > 0$ for some $1 \leqslant l \leqslant r'$ are zero. Thus we have an induced homomorphism of algebras

$$Z_k \to H(\hat{A}_k).$$

5.28. THEOREM. *The subalgebra $Z_k \subset \hat{A}_k$ induces an isomorphism*

$$Z_k \cong H(\hat{A}_k),$$

i.e. the monomial cocycles $z_{(i,j)}$ satisfying (5.25) *to* (5.27) *form a linear basis of $H(\hat{A}_k)$.*

PROOF. We filter the complex $\hat{A}_k$ by the graded subspaces $X^{\cdot}_{(m)} = \amalg_{s=0}^{r'} X_{(m)}^{-s}$ with $X_{(m)}^{-s} = \Lambda_s(y_1, \dots, y_m) \otimes K[c_1, \dots, c_r]_k$, and denote $Z_{(m)}^{-s} = Z^{-s} \cap X_{(m)}^{-s}$. Clearly $X_{(m)}$ is a differential filtration of $\hat{A}_k$ satisfying $X_{(0)} = K[c_1, \dots, c_r]_k$, $X_{(m)} \subset X_{(m+1)}$, $X_{(r')} = \hat{A}_k$ and $X_{(m)}^{-s} \neq 0$ only for $0 \leq s \leq m$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to H^{-s}(X^{\cdot}_{(m-1)}) & \xrightarrow{(\cdot c_{\alpha_m})_*} & H^{-s}(X^{\cdot}_{(m-1)}) & \xrightarrow{i_*} & H^{-s}(X^{\cdot}_{(m)}) & \xrightarrow{(j_m)_*} & H^{-s+1}(X^{\cdot}_{(m-1)}) & \to \cdots & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots \to Z_{(m-1)}^{-s} & \xrightarrow{\cdot c_{\alpha_m}} & Z_{(m-1)}^{-s} & \xrightarrow{i} & Z_{(m)}^{-s} & \xrightarrow{j_m} & Z_{(m-1)}^{-s+1} & \to \cdots & (5.29)
\end{array}$$

where $\cdot c_{\alpha_m}$ denotes multiplication with c_{α_m} and the maps i and the vertical maps are given by inclusion. The map j_m: $X_{(m)}^{-s} \to X_{(m-1)}^{-s+1}$ is determined on monomials by

$$j_m(y_{i_1} \wedge \cdots \wedge y_{i_{s-1}} \wedge y_m \otimes c_{(j)}) = y_{i_1} \wedge \cdots \wedge y_{i_{s-1}} \otimes c_{(j)}$$

if a factor y_m occurs and by 0 otherwise. The top-sequence is exact by standard arguments from the homology theory of rings [CE, Chapter VIII, §4]. Using conditions (5.26) and (5.27) it is immediately verified that the bottom sequence is also exact. Clearly $Z_{(m)}^{0} \cong H^0(X^{\cdot}_{(m)}) \cong K[c_1, \dots, c_r]_k/(c_{\alpha_1}, \dots, c_{\alpha_m})$. By induction over a lexiographical ordering of the pairs (s, m), $0 \leq s \leq m$, and by the 5-lemma we conclude from (5.29) that

$$Z_{(m)}^{-s} \cong H^{-s}(X^{\cdot}_{(m)})$$

for $0 \leq s \leq m$, $0 \leq m \leq r'$. In particular

$$Z_k \cong H(X^{\cdot}_{(r')}) = H(\hat{A}_k)$$

as was to be proved. □

If the pair $(\mathfrak{g}, \mathfrak{h})$ satisfies condition (5.6), we can invoke part (iii) of Theorem 5.13. Together with Theorem 5.28 we conclude that the higher differentials d_i, $i \geq 2$, in the spectral sequence (5.16) are zero and obtain the following result.

5.30 THEOREM. *Let $(\mathfrak{g}, \mathfrak{h})$ be a reductive pair of Lie algebras satisfying condition* (5.6). *Then there is an isomorphism of graded algebras*

$$H(W(\mathfrak{g}, \mathfrak{h})_k) \cong \mathrm{Tor}_{S(\check{V})}(I(\mathfrak{h}), Z_k). \tag{5.31}$$

In particular, if $(\mathfrak{g}, \mathfrak{h})$ is a CS-pair, we have

$$H(W(\mathfrak{g}, \mathfrak{h})_k) \cong Z_k \otimes_{S(\check{V})} I(\mathfrak{h}). \tag{5.32}$$

In the applications of §§6 and 7 we consider only examples of CS-pairs. It might be interesting to find geometric applications involving non-CS-pairs, whose higher Tor-classes in (5.31) are realized in a nontrivial way.

6. Examples. In this section we return to G-foliations of codimension q and discuss their characteristic classes. If the normal bundle Q is equipped with an H-reduction, the characteristic homomorphism is by Theorem 3.6 a map

$$\Delta(Q)_*: H(W(\mathfrak{g}, H)_q) \to H_{\mathrm{DR}}(M). \tag{6.1}$$

If Q admits a basic connection, then the truncation index q can be replaced by $[q/2]$.

Since every G-foliation of codimension q is also a $GL(q)$-foliation, there is by the functoriality of Δ_* a commutative diagram

$$\begin{tikzcd} H(W(\mathfrak{gl}(q), H)_q) \arrow[d, "W(\rho)"'] \arrow[dr] & \\ H(W(\mathfrak{g}, H)_q) \arrow[r] & H_{\mathrm{DR}}(M) \end{tikzcd} \tag{6.2}$$

where ρ^* is induced by ρ: $G \subset GL(q)$. While the kernel of ρ^* consists of classes which are trivial for all G-foliations (equipped with an H-reduction), the classes not in the image of ρ^* are genuinely new for G-foliations.

In all examples below, the pair $(\mathfrak{g}, \mathfrak{h})$ is reductive and satisfies the condition CS, i.e. one of the equivalent conditions of Lemma 5.9. The component group Γ of H is assumed to be finite and then we have the situation explained in formula (3.7), i.e.

$$H(W(\mathfrak{g}, H)_k) \cong H(W(\mathfrak{g}, \mathfrak{h})_k)^{\Gamma}. \tag{6.3}$$

This reduces the problem of the computation of $H(W(\mathfrak{g}, H)_k)$ to the purely algebraic problem discussed in §5. The algorithm of §5 is summarized as follows. The restriction homomorphism $I(\mathfrak{g}) \to I(\mathfrak{h})$ determines the Samelson space $\hat{P}$ of $(\mathfrak{g}, \mathfrak{h})$. Let $\hat{A}_k$ be defined by (5.2) resp. (5.22). By (5.21) then

$$H(W(\mathfrak{g}, \mathfrak{h})_k) \cong H(\hat{A}_k) \otimes_{I(\mathfrak{g})} I(\mathfrak{h}).$$

Let $Z_k \subset \hat{A}_k$ be the subalgebra defined by the cocycles $z_{(i,j)}$ of (5.24) subject to the conditions (5.25) to (5.27). Then $Z_k \cong H(\hat{A}_k)$ which leads to the formula (5.32)

$$H(W(\mathfrak{g}, \mathfrak{h})_k) \cong Z_k \otimes_{S(\check{V})} I(\mathfrak{h}).$$

This algorithm leads in many cases to an explicit determination of $H(W(\mathfrak{g}, H)_k)$, as we are now going to show.

6.4 $(G, H) = (GL(q), O(q))$. In this case

$$\hat{A}_q = \Lambda(y_1, y_3, \ldots, y_{q'}) \otimes \mathbf{R}[c_1, \ldots, c_q]_q \tag{6.5}$$

where y_i is the primitive generator corresponding to c_i and q' the largest odd integer $\leq q$. This follows from the description of the restriction homomorphism in 4.7. For an alternative calculation of Z_q in this case by

Vey see [GB]. The class $y_1 \otimes c_1^q$ is the celebrated Godbillon-Vey class, for which the first nontrivial realization was given in [GV].

In the absolute case, i.e. $(G, H) = (GL(q), \{e\})$, the relevant complex is

$$\hat{A}_q = \Lambda(y_1, y_2, \ldots, y_q) \otimes \mathbf{R}[c_1, \ldots, c_q]_q \tag{6.6}$$

i.e. all primitive generators y_i occur ($\hat{P} = P$).

6.7 $(G, H) = (GL(q), SO(q))$. The complex $\hat{A}_q$ is still given as in formula (6.5). The difference with the previous case resides in $I(\mathfrak{so}(q))$. By (5.21) we have

$$H\big(W(\mathfrak{gl}(q), SO(q))_q\big) \cong \begin{cases} H(\hat{A}_q) & \text{for } q = 2r-1, \\ H(\hat{A}_q)[e_r]/(c_{2r} - e_r^2) & \text{for } q = 2r \end{cases} \tag{6.8}$$

and the formula

$$H\big(W(\mathfrak{gl}(q), O(q))_q\big) \cong H\big(W(\mathfrak{gl}(q), SO(q))_q\big)^{\mathbf{Z}_2}$$

with the obvious action of $\mathbf{Z}_2$ via e_r in case $q = 2r$, and the trivial action for $q = 2r - 1$. It follows that for both parities of q the LHS is given by $H(\hat{A}_q)$.

6.9 $(G, H) = (GL(q), SL(q))$. From the description of the restriction homomorphism in 4.12 it follows that

$$\hat{A}_q = \Lambda(y_1) \otimes \mathbf{R}[c_1, \ldots, c_q]_q. \tag{6.10}$$

The Godbillon-Vey class $y_1 \otimes c_1^q$ comes already from this complex under the canonical map induced by $(GL(q), SO(q)) \to (GL(q), SL(q))$. It follows that the nontriviality of the Godbillon-Vey class $\Delta(Q)_*(y_1 \otimes c_1^q) \in H_{\mathrm{DR}}^{2q+1}(M)$ for a codimension q foliation with normal bundle Q on M obstructs a $GL(q)$-foliation from being a $SL(q)$-foliation. This is its geometric significance.

6.11 Oriented Riemannian foliations (see 2.1). We assume that the normal bundle Q is trivial, so that $(G, H) = (SO(q), \{e\})$. Since there exists a basic connection in Q, the relevant truncation index is $[q/2]$. Since

$$I(SO(2r-1)) \cong \mathbf{R}[p_1, \ldots, p_{r-1}],$$
$$I(SO(2r)) \cong \mathbf{R}[p_1, \ldots, p_{r-1}, e_r],\ e_r^2 = p_r$$

we have to distinguish cases according to the parity of q.

Let y_i $(i = 1, \ldots, r-1)$ be the primitive generators of degree $4i - 1$ corresponding to the Pontrjagin polynomials p_i and assume $q = 2r - 1$. Then $H(W(\mathfrak{so}(2r-1))_{r-1})$ is the cohomology of the complex

$$\Lambda(y_1, \ldots, y_{r-1}) \otimes \mathbf{R}[p_1, \ldots, p_{r-1}]_{r-1}. \tag{6.12}$$

For $q = 2r$ there is a further primitive generator x of degree $2r - 1$ corresponding to the Pfaffian polynomial e_r. The algebra $H(W(\mathfrak{so}(2r))_r)$ is the cohomology of the complex

$$\Lambda(y_1, \ldots, y_{r-1}, x) \otimes \mathbf{R}[p_1, \ldots, p_{r-1}, e_r]_r. \tag{6.13}$$

6.14 Remark. For an oriented Riemannian foliation with trivial normal bundle the classes arising from the complex $\hat{A}_{[q/2]}$ for the pair $(GL(q), O(q))$ or $(GL(q), SO(q))$ are trivial by functoriality. On the other hand the complex $\hat{A}_{[q/2]}$ for $GL(q)$ is of the form

$$\Lambda(\bar{y}_1, \ldots, \bar{y}_q) \otimes \mathbf{R}[c_1, \ldots, c_q]_{[q/2]}.$$

The primitive generators $y_i \in H(\mathfrak{gl}(q))$ corresponding to the c_i's are barred to distinguish them from the y_i corresponding to the p_i's in (6.12). But clearly ρ: $SO(q) \subset GL(q)$ induces the map

$$\rho^*\bar{y}_{2i-1} = 0, \quad \rho^*\bar{y}_{2i} = y_i, \quad \rho^*c_{2i-1} = 0, \quad \rho^*c_{2i} = p_i.$$

Thus ρ^*: $H(W(\mathfrak{gl}(q))_{[q/2]}) \to H(W(\mathfrak{so}(q))_{[q/2]})$ has a large kernel, and is certainly not surjective for $q = 2r$.

6.15 $SL(q)$-FOLIATIONS (see 2.4). For the pair $(SL(q), SO(q))$ we have

$$\hat{A}_q = \Lambda(y_3, y_5, \ldots, y_{q'}) \otimes \mathbf{R}[c_2, \ldots, c_q]_q \tag{6.16}$$

where q' is the largest odd integer $\leqslant q$, and by (5.21)

$$H\big(W(\mathfrak{sl}(q), SO(q))_q\big) \cong \begin{cases} H\big(\hat{A}_q\big) \text{ for } q = 2r - 1, \\ H\big(\hat{A}_q\big)[e_r]/\big(c_{2r} - e_r^2\big) \text{ for } q = 2r. \end{cases} \tag{6.17}$$

For the pair $(SL(q), \{e\})$, i.e. $SL(q)$-foliations with trivial normal bundle, $H(W(\mathfrak{sl}(q))_q)$ is the cohomology of the complex

$$\Lambda(y_2, y_3, \ldots, y_q) \otimes \mathbf{R}[c_2, \ldots, c_q]_q. \tag{6.18}$$

6.19 CONFORMAL FOLIATIONS (see 2.7). For the pair $(C(q), O(q))$ we have

$$\hat{A}_q = \Lambda(z) \otimes \mathbf{R}[p_1, \ldots, p_{[q/2]}] \otimes \mathbf{R}[c]. \tag{6.20}$$

Here $I(\mathfrak{c}(q)) \cong I(\mathfrak{so}(q)) \otimes \mathbf{R}[c]$ and z is the suspension of the Chern polynomial c given by the trace function.

6.21 ALMOST SYMPLECTIC FOLIATIONS (see 2.9). These are foliations of codimension $2q$. To evaluate $H(W(\mathfrak{sp}(q), U(q))_{2q})$ we need to describe the restriction homomorphism $I(\mathfrak{sp}(q)) \to I(\mathfrak{u}(q))$. Note first that $I(\mathfrak{u}(2q)) \cong \mathbf{R}[\tilde{c}_1, \ldots, \tilde{c}_{2q}]$ with the normalization for the Chern polynomials $\tilde{c}_j$ as in [KT 9, 6.20]. The symplectic Pontrjagin polynomials e_j in $I(\mathfrak{sp}(q)) \cong \mathbf{R}[e_1, \ldots, e_q]$ are given by the formulas $e_j = (-1)^j i^*\tilde{c}_{2j}$, where i^* is the restriction homomorphism associated to the inclusion $\mathfrak{sp}(q) \subset \mathfrak{u}(2q)$ (and $i^*\tilde{c}_{2j-1} = 0$). With these notations the map

$$\rho^*: I(\mathfrak{sp}(q)) \cong \mathbf{R}[e_1, \ldots, e_q] \to I(\mathfrak{u}(q)) \cong \mathbf{R}[\tilde{c}_1, \ldots, \tilde{c}_q]$$

is then characterized by

$$\rho^*e_j = \sum_{i+j=2k} (-1)^k \tilde{c}_i \tilde{c}_k.$$

ρ^* is injective, which implies $\hat{P} = 0$ and

$$\hat{A}_{2q} = \mathbf{R}[e_1, \ldots, e_{2q}]_{2q}. \tag{6.22}$$

From (5.21) it follows finally that

$$H\big(W(\mathfrak{sp}(q), U(q))_{2q}\big) \cong \mathbf{R}[\tilde{c}_1, \ldots, \tilde{c}_q] / \Big(\sum_{i+k=2j} (-1)^k \tilde{c}_i \tilde{c}_k \Big)^{\deg > 2q}_{j=1,\ldots,q} \tag{6.23}$$

where the divisor on the RHS is the ideal generated by the polynomials of degree $> 2q$ of the form indicated.

For almost symplectic foliations with trivial normal bundle the cohomology $H(W(\mathfrak{sp}(q))_{2q})$ is the cohomology of the complex

$$\Lambda(y_1, \ldots, y_q) \otimes \mathbf{R}[e_1, \ldots, e_q]_{2q} \tag{6.24}$$

where the y_i are the primitive generators corresponding to the e_i.

6.25 Almost complex foliations (see 2.13). Here the groundfield is $\mathbf{C}$ and $G = GL(q, \mathbf{C})$. The truncation index is the real codimension $2q$. Let $G_{\mathbf{R}}$ denote the underlying real Lie group of $G = GL(q, \mathbf{C})$ with real Lie algebra $\mathfrak{g}_{\mathbf{R}}$. Then $H(W(\mathfrak{gl}(q, \mathbf{C})_{\mathbf{R}}, U(q))_{2q} \otimes \mathbf{C})$ is the cohomology of

$$\hat{A}_{2q} = \Lambda(y_1'', \ldots, y_q'') \otimes \mathbf{C}[c_1', \ldots, c_q'; c_1'', \ldots, c_q'']_{2q}. \tag{6.26}$$

To explain the notations, let $\mathfrak{g}_{\mathbf{R}} \otimes \mathbf{C} \cong \mathfrak{g}^+ \oplus \mathfrak{g}^-$, so that over $\mathbf{C}$

$$I(\mathfrak{g}_{\mathbf{R}} \otimes \mathbf{C}) \cong I(\mathfrak{g}^+) \otimes I(\mathfrak{g}^-) = \mathbf{C}[c_1^+, \ldots, c_q^+; c_1^-, \ldots, c_q^-].$$

Define

$$c_j' = \frac{1}{2}(c_j^+ + c_j^-), \qquad c_j'' = \frac{1}{2i}(c_j^+ - c_j^-) \tag{6.27}$$

and let y_j', y_j'' be the primitive generators corresponding to c_j', c_j'' respectively. Since under the restriction $I(\mathfrak{gl}(q, \mathbf{C})_{\mathbf{R}}) \to I(\mathfrak{u}(q))$ the c_j'' vanish and generate the kernel, the formula (6.26) follows.

If the almost complex structure on the normal bundle Q is integrable (characterized by the vanishing of the relative Eckmann-Frölicher-Nijenhuis tensor N of (2.16)), the characteristic homomorphism is defined on $H((W_{\mathbf{C}}(\mathfrak{g}^+ \otimes \mathfrak{g}^-)_{q,q})_{U(q)})$, the cohomology of the complex

$$\hat{A}_{q,q} = \Lambda(y_1'', \ldots, y_q'') \otimes \mathbf{C}[c_1^+, \ldots, c_q^+]_q \otimes \mathbf{C}[c_1^-, \ldots, c_q^-]_q.$$

These classes measure the incompatibility of the foliated structure of the complex frame bundle of Q with a Hermitian metric on Q.

6.29. For G-foliations it is desirable to incorporate also the affine structure into the construction of characteristic classes. This has been done by Duchamp in [D]. Consider the affine framebundle of a G-foliation with structural group $A_G(\mathbf{R}^q) = G \times \mathbf{R}^q$ (semidirect product). It is a foliated bundle and thus the characteristic homomorphism is well defined. The Weil algebra $W(\mathfrak{a}_G(\mathbf{R}^q))$ is the semidirect product of $W(\mathfrak{g})$ and $W(\mathbf{R}^q)$. The Weil homomorphism of an adapted affine connection involves on the factor $W(\mathfrak{g})$ the connection form and its curvature, and on the factor $W(\mathbf{R}^q)$ the torsion. The Weil homomorphism has then additional truncation properties allowing the definition of additional characteristic invariants. This method is particularly successful for G-foliations defined by tensors and leads to a variety of new characteristic classes for such foliations.

A typical application to oriented Riemannian manifolds is as follows. There is an invariant, $\Delta_*(K_q)$, expressible in terms of the curvature and local framings of M, which gives information about complementary foliations on M. Suppose $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are two transversal, oriented, complementary foliations on M and $\mathfrak{F}_1$ is a regular, codimension-$2q$ foliation with flat leaves. Then all leaves of $\mathfrak{F}_1$ have the same volume, $\mathrm{vol}(\mathfrak{F}_1)$, and the following formula holds **[D]**:

$$\frac{(-1)^q}{(4\pi)^q}\int_M \Delta(K_q) = \mathrm{vol}(\mathfrak{F}_1)\cdot\chi(M/\mathfrak{F}_1)$$

where $\chi(M/\mathfrak{F}_1)$ is the Euler characteristic of the leaf space $M/\mathfrak{F}_1$.

7. Homogeneous foliations. In this section we discuss characteristic classes of homogeneous G-foliations. For the foliation of a Lie group $\bar{G}$ by the left cosets of a subgroup $G \subset \bar{G}$ the normal bundle Q_G is trivial (see 2.19). The characteristic homomorphism is therefore a map

$$\Delta(Q_G)_*\colon H\big(W(\mathfrak{g})_q\big) \to H_{\mathrm{DR}}(\bar{G}),$$

where the codimension q equals $\dim \bar{\mathfrak{g}}/\mathfrak{g}$. The computation of $\Delta(Q_G)_*$ reduces then to a purely Lie algebraic problem **[KT 11]**. For compact G the coset foliation is Riemannian and the truncation index can be replaced by $[q/2]$. On a purely Lie algebraic level the property giving rise to this stronger truncation is the reductivity of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$.

Consider e.g. the case $(\bar{G}, G) = (SO(2r+1), SO(2r))$. Here $q = 2r$ is the dimension of the quotient sphere. Let $\bar{y}_1, \ldots, \bar{y}_r$ be the primitive generators of $H(SO(sr+1))$ corresponding to the Pontrjagin polynomials in $I(\mathfrak{so}(2r+1))$. Then we have the following result.

7.1 THEOREM **[KT 9, 10]**, **[KT 11, 6.52]**. *Let $Q_{SO(2r)}$ be the normal bundle of the foliation of $SO(2r+1)$ by the left cosets of $SO(2r)$ with quotient S^{2r}. The image of the generalized characteristic homomorphism*

$$\Delta_*(Q_{SO(2r)})\colon H\big(W(\mathfrak{so}(2r)_r)\big) \to H(SO(2r+1)) \cong \Lambda(\bar{y}_1, \ldots, \bar{y}_r)$$

is spanned by the linearly independent classes

$$\Delta_*(z_{(i)}) = \bar{y}_{i_1}\wedge\cdots\wedge\bar{y}_{i_s}\wedge\bar{y}_r, \tag{7.1}$$

where $z_{(i)} = y_{i_1}\wedge\cdots\wedge y_{i_s}\wedge x\otimes e_r \in W(\mathfrak{so}(2r))_r$ for $1 \leqslant i_1 < \cdots < i_s \leqslant r-1$, $0 \leqslant s \leqslant r-1$, and

$$\Delta_*(y_{k_1}\wedge\cdots\wedge y_{k_\alpha}\otimes 1) = \bar{y}_{k_1}\wedge\cdots\wedge\bar{y}_{k_\alpha},$$

where $[r/2]+1 \leqslant k_1 < \cdots < k_\alpha \leqslant r-1$. In particular

$$\mathrm{im}\,\Delta_*(Q_{SO(2r)}) = \mathrm{Id}(\bar{y}_r) \oplus \Lambda\big(\bar{y}_{[r/2]+1}, \ldots, \bar{y}_{r-1}\big) \subset H(SO(2r+1))$$

and thus

$$\dim\big(\mathrm{im}\,\Delta_*\big) = 2^{r-1} + 2^{[(r-1)/2]}.$$

It follows that $\Delta(Q_{SO(2r)})_*$ is surjective if and only if $r = 1$.

Another example is the case $(\bar{G}, G) = (SU(r+1), U(r))$. Here $U(r)$ is realized in $SU(r+1)$ by the matrices

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & \lambda^{-1} \end{array}\right]$$

with $A \in U(r)$, $\lambda = \det A$. The target of the characteristic homomorphism is

$$H(SU(r+1)) \cong \Lambda(\bar{y}_2, \ldots, \bar{y}_{r+1}),$$

where the primitive generators $\bar{y}_2, \ldots, \bar{y}_{r+1}$ correspond to the Chern polynomials $\bar{c}_2, \ldots, \bar{c}_{r+1}$ in $I(\mathfrak{su}(r+1))$. The algebra $H(W(\mathfrak{u}(r))_r)$ is realized by the complex

$$\Lambda(y_1, \ldots, y_r) \otimes \mathbf{R}[c_1, \ldots, c_r]_r$$

in the by now standard fashion. The following result holds in this case.

7.2 THEOREM [KT 9], [KT 10], [KT 11, 6.49]. *Let $Q_{U(r)}$ be the foliated complex normal bundle of the foliation of $SU(r+1)$ defined by the right action of $U(r)$ with quotient space $\mathbf{P}^r\mathbf{C}$. Then the image of the generalized characteristic homomorphism*

$$\Delta_*^+(Q_{U(r)}): H^+(W(\mathfrak{u}(r))_r) \to H^+(SU(r+1)) \cong \Lambda^+(\bar{y}_2, \ldots, \bar{y}_{r+1})$$

is spanned by the linearly independent classes

$$\Delta_*(z_{(i)}) = \kappa \bar{y}_{i_1} \wedge \cdots \wedge \bar{y}_{i_s} \wedge \bar{y}_{r+1},$$

where $z_{(i)} = y_1 \wedge y_{i_1} \wedge \cdots \wedge y_{i_s} \otimes c_1^r \in W(\mathfrak{u}(r))_r$ for $2 \leqslant i_1 < \cdots < i_s \leqslant r$, $0 \leqslant s \leqslant r-1$ and $\kappa = (-1)^{s+1} \cdot (r+1)^{r+1}$. In particular

$$\operatorname{im} \Delta_*^+(Q_{U(r)}) = \operatorname{Id}(\bar{y}_{r+1}) \subset H^+(SU(r+1)).$$

It follows that

$$\dim \operatorname{im} \Delta(Q_{U(r)})_*^+ = 2^{r-1}.$$

Note that $\dim H(SU(r+1)) = 2^r$.

Further example of characteristic classes for Riemannian foliations have been calculated by Lazarov-Pasternack [LP 1].

Another interesting class of foliations is obtained in the following way. We begin with a connected semisimple Lie group $\bar{G}$ with finite center and no compact factor, and a maximal compact subgroup $K_{\bar{G}}$. For a discrete uniform and torsion free subgroup $\Gamma \subset \bar{G}$ we have then the flat bundle

$$\bar{P} = (\Gamma \setminus \bar{G}) \times_{K_{\bar{G}}} \bar{G} \cong \bar{G}/K_{\bar{G}} \times_\Gamma \bar{G} \tag{7.3}$$

on the Clifford-Klein form $X = \Gamma \setminus \bar{G}/K_{\bar{G}}$ of the noncompact symmetric space $\bar{G}/K_{\bar{G}}$ (4.23). Observe that the isomorphism in (7.3) is induced by the mapping $\varphi: \bar{G} \times \bar{G} \to \bar{G} \times \bar{G}$ defined by $\varphi(g, g') = (g, gg')$, $g, g' \in \bar{G}$.

Let $G \subset \bar{G}$ be a closed connected subgroup with maximal compact subgroup K_G. Assume that the canonical map

$$K_{\bar{G}}/K_G \xrightarrow{\cong} \bar{G}/G \tag{7.4}$$

is an isomorphism. Then the foliated G-bundle $\bar{P} \to \bar{P}/G = M$ is isomorphic to

$$P = \Gamma \setminus \bar{G} \times_{K_G} G \to \Gamma \setminus \bar{G}/K_G. \tag{7.5}$$

We are in the particular case of the situation discussed in 2.23 where $\bar{P} \to X$ is a flat bundle. By (7.3), (7.4) the space M is given by

$$M \cong \bar{G}/K_{\bar{G}} \times_\Gamma \bar{G}/G \cong \Gamma \setminus \bar{G}/K_G.$$

The quotient foliation on M of the foliation on $\bar{P}$ defined by its flat structure is transverse to the fiber $\bar{G}/G$ of $M \to X$. Its normal bundle Q_G is associated to $P \xrightarrow{\hat{\pi}} M$ via the isotropy representation ρ of G in $\mathfrak{m} = \bar{\mathfrak{g}}/\mathfrak{g}$:

$$Q_G \cong T(\hat{\pi}) \cong P \times_G \mathfrak{m}_\rho.$$

Note that the characteristic homomorphism $\Delta(\bar{P}) = \bar{s}^* \circ k(\bar{\omega})_{K_{\bar{G}}}$ of the flat bundle $\bar{P}$ is given by the canonical inclusion $\bar{\gamma}$ of the left invariant forms $(\Lambda\bar{\mathfrak{g}}^*)_{K_{\bar{G}}}$ into the De Rham complex of $X = \Gamma \setminus \bar{G}/K_{\bar{G}}$. The characteristic homomorphism $\Delta(Q_G)$ on the cochain level appears then in the following commutative diagram:

$$\begin{tikzcd}
W(\mathfrak{gl}(q), SO(q))_q \arrow[d, "W(\rho)"'] \arrow[ddr, "\Delta(Q_G)"] & \\
W(\mathfrak{g}, K_G)_q \arrow[d, "\Delta(\theta)"'] \arrow[dr, "\Delta(P)"] & \\
(\Lambda\bar{\mathfrak{g}}^*)K_G \arrow[r, "\gamma"] \arrow[d, "j_*"'] & \Omega(\Gamma\setminus\bar{G}/K_G) \arrow[d, "\hat{\pi}_*"] \\
(\Lambda\bar{\mathfrak{g}}^*)K_{\bar{G}} \arrow[r, "\bar{\gamma}"] & \Omega(\Gamma\setminus\bar{G}/K_{\bar{G}})
\end{tikzcd} \tag{7.6}$$

In this diagram, $\Delta(P)$ realizes the characteristic homomorphism of the foliated G-bundle P with its canonical K_G-reduction. Q_G is associated to P via the isotropy representation ρ: $\mathfrak{g} \to \mathfrak{gl}(\bar{\mathfrak{g}}/\mathfrak{g})$, where $q = \dim \bar{\mathfrak{g}}/\mathfrak{g}$. $W(\rho)$ is the induced map on Weil algebras. γ and $\bar{\gamma}$ are induced by the canonical inclusion $\Lambda\bar{\mathfrak{g}}^* \to \Omega(\bar{G})$ and $\bar{\gamma} = \Delta(\bar{P})$. The maps j_* and $\hat{\pi}_*$ denote integration over the fiber $K_{\bar{G}}/K_G$ in the respective fibrations on the left (algebraic) and the right (geometric). $\Delta(\theta)$ finally is induced by a K_G-invariant splitting θ: $\bar{\mathfrak{g}} \to \mathfrak{g}$ of the exact sequence $0 \to \mathfrak{g} \to \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\mathfrak{g} \to 0$. It is a map completely characterized by

$$\begin{aligned}
&\Delta(\theta)\alpha = \alpha\theta \quad \text{for } \alpha \in \Lambda^1\mathfrak{g}^*, \\
&\Delta(\theta)\tilde{\alpha} = \alpha K(\theta) = d\alpha\theta + \tfrac{1}{2}\alpha[\theta, \theta] \quad \text{for } \tilde{\alpha} \in S^1\mathfrak{g}^*.
\end{aligned} \tag{7.7}$$

The computation of $\Delta(Q_G)_*$ reduces to the computation of the cohomology maps induced by the algebraically defined maps $W(\rho)$, $\Delta(\theta)$ by virtue of the following result.

7.8 THEOREM ([KT 9]–[KT 11]). *Let $\bar{G}$, G and Γ be as above. Let q be the codimension of the canonical G-foliation on $\Gamma \setminus \bar{G}/K_G$, with normal bundle Q_G, $q = \dim \bar{\mathfrak{g}}/\mathfrak{g}$. Let θ: $\bar{\mathfrak{g}} \to \mathfrak{g}$ be a K_G-equivariant splitting of the exact sequence*

$$0 \to \mathfrak{g} \to \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\mathfrak{g} \to 0. \tag{7.9}$$

Then the generalized characteristic homomorphism $\Delta(Q_G)$ on the cochain level factorizes as in the commutative diagram (7.6). The map $\Delta(\theta)$ is characterized by (7.7). The maps induced by γ and $\bar{\gamma}$ in cohomology

$$\gamma_*: H(\bar{\mathfrak{g}}, K_G) \to H_{\mathrm{DR}}(\Gamma \setminus \bar{G}/K_G),$$
$$\bar{\gamma}_*: H(\bar{\mathfrak{g}}, K_{\bar{G}}) \to H_{\mathrm{DR}}(\Gamma \setminus \bar{G}/K_{\bar{G}})$$

are both injective.

A typical application is given by the pair

$$\bar{G} = SL(q+1) \text{ and } G = SL(q+1, 1)_0,$$

the connected component of the group of unimodular matrices of the form

$$\left[\begin{array}{c|c} \lambda & * \\ \hline 0 & A \end{array}\right]$$

with $A \in GL(q)$, $\det A = \lambda^{-1}$. Some of the computations for this example have been carried out independently by Shulman and Tischler [ST].

7.10 THEOREM [KT 12, 5.37]. *Consider the spherical fibration*

$$M = \Gamma \setminus SL(q+1)/SO(q) \xrightarrow{\hat{\pi}} X = \Gamma \setminus SL(q+1)/SO(q+1)$$

over the Clifford-Klein form X of the symmetric space $SL(q+1)/SO(q+1)$. Then

$$M \cong SL(q+1)/SO(q+1) \times_\Gamma SL(q+1)/SL(q+1, 1)_0$$

carries a foliation of codimension q defined either by the right-action of $SL(q+1, 1)_0$ on $SL(q+1)$ or by the flat structure of $\hat{\pi}$: $M \to X$. This foliation is transverse to the fiber S^q and every leaf is a universal covering space of X under the projection $\hat{\pi}$. The normal bundle Q_G of this foliation ($G = SL(q+1, 1)_0$) with its natural foliated structure is given by

$$\begin{aligned} Q_G &= T(\hat{\pi}) = \Gamma \setminus SL(q+1) \times_{SO(q)} \mathbf{R}^q \\ &\cong (\Gamma \setminus SL(q+1) \times_{SO(q)} G) \times_G \mathfrak{m}_\rho, \end{aligned}$$

where $\mathfrak{m}_\rho \cong \mathbf{R}^q$ is equipped with the action ρ: $SL(q+1, 1)_0 \to GL(q)$ sending

$$\left[\begin{array}{c|c} \lambda & * \\ \hline 0 & A \end{array}\right]$$

to $\lambda^{-1}A$, $\lambda = \det A^{-1}$.

The characteristic homomorphism

$$\Delta(Q_G)_*: H\big(W(\mathfrak{gl}(q), SO(q))_q\big) \to H_{\mathrm{DR}}(M)$$

of this foliation has then the following properties (see diagram (7.6) for the maps involved).

(i) *Consider in $H(W(\mathfrak{gl}(q), SO(q))_q)$ the classes of the cocycles*

$$z_{(i,j)} = y_1 \wedge y_{2i_1-1} \wedge \cdots \wedge y_{2i_s-1} \otimes c_{(j)},$$

where $2 \leq i_1 < \cdots < i_s \leq r$, $0 \leq s \leq r-1$; $c_{(j)} = c_1^{j_1} \cdots c_q^{j_q}$ *with* $\deg c_{(j)} = 2\sum_{l=1}^{q} j_l \cdot l = 2q$ ($z_{(\emptyset,j)} = y_1 \otimes c_{(j)}$ *for* $s = 0$). *Then in* $H_{\mathrm{DR}}(M)$

$$\Delta(Q_G)_*[z_{(i,j)} \otimes \Phi] = \mu \cdot \gamma_*(\bar{y}_{2i_1-1} \wedge \cdots \wedge \bar{y}_{2i_s-1} \wedge \bar{y}_{q+1} \otimes \Phi),$$

where $\Phi = 1$ *for* $q = 2r-1$ *and* $\Phi = 1$ *or* e_r *for* $q = 2r$, *and* $\mu = (-1)^{s+1}(q+1)\prod_{l=1}^{q}\binom{q+1}{l}^{j_l}$.

(ii) *Let* $q = 2r - 1$. *Then*

$$\operatorname{im} \Delta(Q_G)_*^+ = \gamma_*(\bar{y}_{2r} \cdot \Lambda(\bar{y}_3, \ldots, \bar{y}_{2r-1}))$$

and $\Delta(Q_G)_*$ *maps the cocycles*

$$y_1 \wedge y_{2i_1-1} \wedge \cdots \wedge y_{2i_s-1} \otimes c_q$$

with $2 \leq i_1 < \cdots < i_s \leq r$, $0 \leq s \leq r-1$ *onto an* $\mathbf{R}$-*basis of* $\operatorname{im} \Delta(Q_G)_*^+ \subset H_{\mathrm{DR}}^+(M)$. *It follows that*

$$\dim \operatorname{im} \Delta(Q_G)_*^+ = 2^{r-1}.$$

(iii) *Let* $q = 2r$. *Then*

$$\operatorname{im} \Delta(Q_G)_*^+ = \gamma_*(\bar{y}_{2r+1} \cdot \Lambda(\bar{y}_3, \ldots, \bar{y}_{2r-1}) \otimes \mathbf{R}[e_r]/(e_r^2)),$$

and $\Delta(Q_G)_*$ *maps the cocycles* $y_1 \wedge y_{2i_1-1} \wedge \cdots \wedge y_{2i_s-1} \otimes c_q$ *and* $y_1 \wedge y_{2i_1-1} \wedge \cdots \wedge y_{2i_s-1} \otimes c_q \otimes e_r$ *with* $2 \leq i_1 < \cdots < i_s \leq r$, $0 \leq s \leq r-1$ *onto an* $\mathbf{R}$-*basis of* $\operatorname{im} \Delta(Q_G)_*^+ \subset H_{\mathrm{DR}}^+(M)$. *It follows that*

$$\dim \operatorname{im} \Delta(Q_G)_*^+ = 2^r.$$

Concerning the integration over the fiber $\hat{\pi}_*$ *we have the following results.*

(iv) *For* $q = 2r - 1$

$$\hat{\pi}_*\Delta(Q_G)_*[z_{(i,j)}] = \mu \cdot \langle \sigma e_r, S^{2r-1} \rangle \cdot \bar{\gamma}_*(\bar{y}_{2i_1-1} \wedge \cdots \wedge \bar{y}_{2i_s-1} \cdot e_r)$$

where μ *is as in* (i), $\bar{\gamma}_*$ *is injective,*

$$\langle \sigma e_r, S^{2r-1} \rangle = (-1)^r / 2^{2(r-1)}$$

and $\bar{\gamma}_*(e_r)$ *is the Euler class of the flat* $SL(2r)$-*bundle* $\bar{P} \to X$.

(v) *For* $q = 2r$

$$\hat{\pi}_* \Delta(Q_G)_*[z_{(i,j)}] = 0 \quad \textit{and}$$

$$\hat{\pi}_* \Delta(Q_G)_*[z_{(i,j)} \otimes e_r] = \mu \cdot \langle e_r, S^{2r} \rangle \cdot \bar{\gamma}_*(\bar{y}_{2i_1-1} \wedge \cdots \wedge \bar{y}_{2i_s-1} \wedge \bar{y}_{2r+1} \otimes 1)$$

where μ *is as in* (i), $\bar{\gamma}_*$ *is injective and* $\langle e_r, S^{2r} \rangle = 2$.

7.11 COROLLARY. *Let* $q = 2r - 1$. *For the Godbillon-Vey class* $y_1 \otimes c_1^q \in H(W(\mathfrak{gl}(q), SO(q))_q)$ *we have by* (i) *and* (iv) *of the preceding theorem the formulas*

$$\Delta(Q_G)_*(y_1 \otimes c_1^q) = -(2r)^{2r}\gamma_*(\bar{y}_{2r})$$

and

$$\hat{\pi}_*\Delta(Q_G)_*(y_1 \otimes c_1^q) = -(2r)^{2r} \cdot \langle \sigma e_r, S^{2r-1} \rangle \cdot \bar{\gamma}_*(e_r).$$

Here $\bar{\gamma}_*(e_r)$ *is the nonzero Euler class of the flat* $SL(2r)$-*bundle*

$$SL(2r)/SO(2r) \times_\Gamma SL(2r) \to \Gamma \setminus SL(2r)/SO(2r).$$

For $q = 1$ in particular the integration map $\hat{\pi}_*$: $H^3(M) \to H^2(X) \cong \mathbf{R}$ is an isomorphism and the Godbillon-Vey class satisfies the formula

$$\langle \hat{\pi}_* \Delta(Q_G)_*(y_1 \otimes c_1), [X] \rangle = 4\langle \bar{\gamma}_*(e_r), [X] \rangle. \tag{7.12}$$

As the Euler number $\langle \bar{\gamma}_*(e_r), [X] \rangle$ remains constant under deformations of Γ in $SL(2)$, it follows that $\Delta(Q_G)_*(y_1 \otimes c_1)$ satisfies the same property.

We wish to take this opportunity to point out an error in **[KT 9**, Remark 7.28], which is obviously contradicted by the last statement above. The formula for $\langle \sigma e_r, S^{2r-1} \rangle$ in part (iv) of 7.10 was misstated in **[KT 9]**, **[KT 11]**.

There is a similar, but more elementary result for the foliation obtained before dividing the whole situation by $SO(q+1)$. In that case the projection $M \to X$ is the trivial spherical fibration.

$$\Gamma \setminus SL(q+1) \times SL(q+1)/SL(q+1, 1)_0 \to \Gamma \setminus SL(q+1).$$

To test the nontriviality of cohomology classes in the total space is then easier than in the case of Theorem 7.10. Some of the technical difficulties evaporate for this situation.

We wish to discuss an application to the cohomology $H(B\Gamma_q^+)$ of the classifying space for codimension q foliations with oriented normal bundle. Such a foliation is characterized (up to homotopy) by a map f: $M \to B\Gamma_q^+$. There is then a commutative diagram

$$\begin{tikzcd} & H(B\Gamma_q^+, \mathbf{R}) \arrow[d, "f^*"] \\ H(W(\mathfrak{gl}(q), SO(q))_q) \arrow[ur, "\Delta_*"] \arrow[r, "\Delta(Q)_*"'] & H_{\mathrm{DR}}(M) \end{tikzcd} \tag{7.13}$$

where $\Delta(Q)_*$ denotes the characteristic homomorphism of the (normal bundle Q of the) given foliation on M. This is an obvious consequence of the functoriality of the generalized characteristic homomorphism. At this point we use the fact that the construction of the generalized characteristic homomorphism as given applies to singular Haefliger Γ_q-cocycles. This defines Δ_* by universality. From diagram (7.13) it follows then that the cohomology classes of $H(W(\mathfrak{gl}(q), SO(q))_q)$ which are realized linearly independently under $\Delta(Q)_*$ for one foliation with oriented normal bundle Q, are by necessity also linearly independently realized under the universal map Δ_*. A similar argument holds for the universal map $\tilde{\Delta}_*$: $H(W(\mathfrak{gl}(q))_q) \to H(F\Gamma_q)$ and the generalized characteristic homomorphism $\tilde{\Delta}(q)_*$ of a foliation of codimension q on M with trivial normal bundle Q, classified (up to homotopy) by a map $\tilde{f}$: $M \to F\Gamma_q$. In that case there is again by functoriality a commutative diagram

$$\begin{tikzcd} & H(F\Gamma_q, \mathbf{R}) \arrow[d, "\tilde{f}^*"] \\ H(W(\mathfrak{gl}(q))_q) \arrow[ur, "\tilde{\Delta}_*"] \arrow[r, "\tilde{\Delta}(Q)_*"'] & H_{\mathrm{DR}}(M) \end{tikzcd} \tag{7.14}$$

By Theorem 7.10 and its corresponding absolute version one obtains then the following linear independence results [**KT 12**].

7.15 COROLLARY.

(i) *In* $H(B\Gamma_q^+)$. *Let* $c_{(j)} = c_1^{j_1} \cdots c_q^{j_q}$ *be any monomial of* $\deg c_{(j)} = 2q$. *For* $q = 2r - 1$ *and* $q = 2r$ *the classes of the cocycles in* $W(\mathfrak{gl}(q), SO(q))_q$

$$y_1 \wedge y_{2i_1-1} \wedge \cdots \wedge y_{2i_s-1} \otimes c_{(j)} \tag{7.16}$$

for all $2 \leqslant 1_i < \cdots < i_s \leqslant r$, $0 \leqslant s \leqslant r-1$ *are realized linearly independently in* $H(B\Gamma_q^+)$ *under* Δ_* (*for* $s = 0$ *the cocycle* (7.16) *is* $y_1 \otimes c_{(j)}$). *For* $q = 2r$ *the union of the classes corresponding to* (7.16) *and the cocycles*

$$y_1 \wedge y_{2i_1-1} \wedge \cdots \wedge y_{2i_s-1} \otimes c_{(j)} \otimes e_r \tag{7.17}$$

are realized as a linearly independent set in $H(B\Gamma_q^+)$ *under* Δ_*. *The corresponding cohomology classes span then in particular a subspace of dimension* 2^{r-1} *for* $q = 2r - 1$ *and dimension* 2^r *for* $q = 2r$.

(ii) *In* $H(F\Gamma_q)$. *Let* $c_{(j)} = c_1^{j_1} \cdots c_1^{j_q}$ *be any monomial of* $\deg c_{(j)} = 2q$. *The classes of the cocycles in* $W(\mathfrak{gl}(q))_q$

$$y_1 \wedge y_{i_1} \wedge \cdots \wedge y_{i_s} \otimes c_{(j)} \tag{7.18}$$

for arbitrary $2 \leqslant i_1 < \cdots < i_s \leqslant q$, $0 \leqslant s \leqslant q-1$ *are realized linearly independently in* $H(F\Gamma_q)$ *under* $\tilde{\Delta}_*$. *The corresponding cohomology classes span then in particular a subspace of dimension* 2^{q-1}.

Concerning this result, see also the addendum at the end of this section. Foliations of the type described have also been considered by Fuchs [**F1**] and Baker [**BK**]. In the latter paper $\Delta(Q_G)_*$ is determined for a wide class of such foliations. Linear independence results in the single dimension $m = 2q + 1$ have been established by Morita [**MR**] and Yamato [**Y**]. For many of the classes in this dimension $m = 2q + 1$ it has been shown that they even can vary linearly independently in the following sense. A set of classes $z_1, \ldots, z_d \in H^m(B\Gamma_q)$ vary linearly independently, if the canonical map $\zeta\colon H_m(B\Gamma_q) \to \mathbf{R}^d$ defined by $\zeta(x) = (z_i(x))_{i=1,\ldots,d}$ is surjective. Here the numbers $z_i(x)$ for a cycle x represented by a (possibly singular) foliation $f\colon M^n \to B\Gamma_q$ are given by $z_i(x) = (f^*z_i)[M]$. Heitsch [**HT 3**] has established the linearly independent variation of many of the classes in dimension $m = 2q + 1$. These results generalize Thurston's theorem on the surjectivity of the map $\pi_3(B\Gamma_1) \to \mathbf{R}$ given by the Godbillon-Vey number of codimension 1 foliations on S^3 [**TH**]. Such results for holomorphic foliations have earlier been proved by Bott [**B 2**], and for Riemannian foliations by Lazarov-Pasternack [**LP 2**].

It is of interest to return to the evaluation principle embodied in diagram (7.6). The essential geometric situation is a fibration $\hat{\pi}\colon M \to X$, carrying a foliation transverse to the homogeneous fiber $F \equiv \bar{G}/G \cong K_{\bar{G}}/K_G$. This flat structure is described as in (2.27) by an isomorphism $M \cong \tilde{X} \times_\Gamma F$, where Γ acts on F via a homomorphism $\Gamma = \pi_1(X) \to \bar{G}$.

It is important to allow one to enlarge the group $\bar{G}$ from a Lie group acting on F to any subgroup $\bar{G}$ of the diffeomorphism group $\mathrm{Diff}(F)$ acting transitively on F. We assume that $\bar{G}$ admits a compact subgroup $\bar{K}$ still acting transitively on the compact fiber F. If G and K denote the isotropy groups of $\bar{G}$ and $\bar{K}$ at the basepoint $x_0 \in F$, then $F \cong \bar{K}/K \cong \bar{G}/G$. For a flat principal $\bar{G}$-bundle $\bar{P} \to X$ with a $\bar{K}$-reduction $\bar{K} \to P \to X$ we have then an associated fiber space $M = P \times_{\bar{K}} F \cong P/K \cong \tilde{X} \times_\Gamma F$. The fibration $\hat{\pi}$: $M \to X$ has again two geometric structures, namely a flat structure and a $\bar{K}$-fiberbundle structure. The incompatibility of these structures gives rise to a characteristic homomorphism

$$\bar{\gamma}_*: H_c(\bar{\mathfrak{g}}, \bar{K}) \to H_{\mathrm{DR}}(X)$$

as constructed by Haefliger [H 2]. Here $\bar{\mathfrak{g}}$ denotes the Lie algebra of vectorfields on F defined by the infinitesimal action of $\bar{G}$. For $\bar{G} = \mathrm{Diff}(F)$ this is the Lie algebra $\mathfrak{L}(F)$ of all global vectorfields on F. The $\bar{K}$-basic continuous cochains (in the C^∞-topology) with the usual Chevalley-Eilenberg differential give rise to the Gelfand-Fuchs cohomology $H_c(\bar{\mathfrak{g}}, \bar{K})$ [GF 1], [GF 2], [H 3], [H 4]. Heuristically this cohomology algebra plays for the pair $(\bar{G}, \bar{K})$ the same role as that played by the relative Chevalley-Eilenberg cohomology in the finite-dimensional case. After the first computation of such an algebra $H_c(\mathfrak{L}(F))$ in the case of $F = S^1$ by Gelfand-Fuchs [GF 1], [GF 2], Haefliger has given in [H 3], [H 4] a general structure theorem under certain assumptions on F.

Haefliger's construction of $\bar{\gamma}_*$ reduces in the finite-dimensional case to the map induced by $\bar{\gamma}$ in diagram (7.6). The same contruction also induces a map γ_*: $H_c(\bar{\mathfrak{g}}, K) \to H_{\mathrm{DR}}(M)$. These two maps relate via the following commutative diagram corresponding to the bottom square of (7.6)

$$\begin{tikzcd} H_c(\bar{\mathfrak{g}}, K) \arrow[r, "\gamma_*"] \arrow[d, "j_*"'] & H_{\mathrm{DR}}(M) \arrow[d, "\hat{\pi}^*"] \\ H_c(\bar{\mathfrak{g}}, \bar{K}) \arrow[r, "\bar{\gamma}_*"] & H_{\mathrm{DR}}(X) \end{tikzcd} \tag{7.19}$$

Here j_* and $\hat{\pi}_*$ denote again integration maps over the compact fiber $F \cong \bar{K}/K$.

The observation in the finite-dimensional case embodied in diagram (7.6) is that the characteristic homomorphism of the foliation transverse to the fiber of the flat bundle projection $\hat{\pi}$ factorizes through γ_*. This property holds true for both the construction of the characteristic homomorphism à la Bernstein-Rosenfeld [BR 1], [BR 2] and Bott-Haefliger [BH], [H2] with the domain the Gelfand-Fuchs complex of the Lie algebra $\mathfrak{L}_0(\mathbf{R}^q)$ of formal vectorfields of $\mathbf{R}_q$, and the authors' construction [KT 3], etc., with domain the relative truncated Weil algebra (as used throughout this paper). To compare the two constructions, it is therefore sufficient to compare them as maps into

$H_c(\bar{\mathfrak{g}}, K)$ (the actual characteristic homomorphisms are then the compositions with γ_*).

For the purposes of this comparison let jet: $\mathfrak{L}(F) \to \mathfrak{L}_0(\mathbf{R}^q)$ be the map assigning to any vectorfield on F its formal jet expansion at the basepoint $x_0 \in F$. Here $q = \dim F = \dim \bar{\mathfrak{g}}/\mathfrak{g}$. Let α: $\mathfrak{L}_0(\mathbf{R}^q) \to \mathfrak{gl}(q)$ be defined by the assignment of the linear part to a formal vectorfield. For the subalgebra $\bar{\mathfrak{g}} \subset \mathfrak{L}(F)$ we have similarly maps $\bar{\mathfrak{g}} \xrightarrow{\text{jet}} \mathfrak{L}_0(\bar{\mathfrak{g}}) \xrightarrow{\alpha(\bar{\mathfrak{g}})} \bar{\mathfrak{g}}_1$. The corresponding maps $\mathfrak{g} \xrightarrow{\text{jet}} \mathfrak{L}_0(\mathfrak{g}) \to \mathfrak{g}_1$ for the isotropy algebra $\mathfrak{g}$ at x_0 are Lie homomorphisms. It is easy to see that for the linear parts of $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ we have $\bar{\mathfrak{g}}_1 = \mathfrak{g}_1$. All these maps are related by the commutative diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, phantom, "\subset"] \arrow[d, "\text{jet}"'] & \bar{\mathfrak{g}} \arrow[l, bend right, "\theta"'] \arrow[r, phantom, "\overset{\epsilon}{\subset}"] \arrow[d, "\text{jet}"] & \mathfrak{L}(F) \arrow[d, "\text{jet}"] \\
\mathfrak{L}_0(\mathfrak{g}) \arrow[r] \arrow[d, "\alpha(\mathfrak{g})"] & \mathfrak{L}_0(\bar{\mathfrak{g}}) \arrow[l, bend right, "\mathfrak{L}_0(\theta)"'] \arrow[r, "\mathfrak{L}_0(\epsilon)"] \arrow[d, "\alpha(\bar{\mathfrak{g}})"] & \mathfrak{L}_0(\mathbf{R}^q) \arrow[d, "\alpha"] \\
\mathfrak{g}_1 \arrow[r, phantom, "="] & \bar{\mathfrak{g}}_1 \arrow[r, "\epsilon_1"] & \mathfrak{gl}(q)
\end{tikzcd} \tag{7.20}$$

Note that the counter-clockwise map ρ: $\mathfrak{g} \to \mathfrak{gl}(q) \cong \mathfrak{gl}(\bar{\mathfrak{g}}/\mathfrak{g})$ around this diagram is the isotropy representation.

As before let θ: $\bar{\mathfrak{g}} \to \mathfrak{g}$ be a K-equivariant retraction of the inclusion $\mathfrak{g} \subset \bar{\mathfrak{g}}$ with induced map $\mathfrak{L}_0(\theta)$ on the formal jet expansions. The map $\Delta(\theta)$ of diagram (7.6) has an infinite-dimensional analogue $\Delta(\theta)$: $W_c(\mathfrak{g}, K)_q \to C_c(\bar{\mathfrak{g}}, K)$. Here the complex $W_c(\mathfrak{g})$ denotes the continuous Chevalley-Eilenberg cochains $C_c(\mathfrak{g}, S\mathfrak{g}^*)$ with the inherited Weil differential. Similarly there is a map $\Delta(\mathfrak{L}_0(\theta))$: $W_c(\mathfrak{L}_0(\mathfrak{g}), K)_q \to C_c(\mathfrak{L}_0(\bar{\mathfrak{g}}), K)$ of DG-algebras. Corresponding to (7.20), there is then the following commutative diagram

$$\begin{tikzcd}
H_c(W(\mathfrak{g}, K)_q) \arrow[r, "\Delta(\theta)_*"] & H_c(\bar{\mathfrak{g}}, K) & H_c(\mathfrak{L}(F), K) \arrow[l, "\epsilon^*"'] \\
H_c(W(\mathfrak{L}_0(\mathfrak{g}), K)_q) \arrow[u, "\text{jet}^*"'] \arrow[r, "\Delta(\mathfrak{L}_0(\theta))_*"] & H_c(\mathfrak{L}_0(\bar{\mathfrak{g}}), K) \arrow[u, "\text{jet}^*"'] & H_c(\mathfrak{L}_0(\mathbf{R}^q), O(q)) \arrow[l] \arrow[u, "\text{jet}^*"'] \\
H(W(\mathfrak{g}_1, K)_q) \arrow[u, "W(\alpha(\mathfrak{g}))_*"'] \arrow[r, phantom, "="] & H(W(\bar{\mathfrak{g}}_1, K)_q) \arrow[u, "\Delta(\alpha(\bar{\mathfrak{g}}))_*"'] & H(W(\mathfrak{gl}(q), O(q))_q) \arrow[l] \arrow[u, "\cong", "\Delta(\alpha)_*"']
\end{tikzcd} \tag{7.21}$$

The clockwise map

$$W(\rho)_*: H\big(W(\mathfrak{gl}(q), O(q))_q\big) \to H_c\big(W(\mathfrak{g}, K)_q\big)$$

from the lower right to the upper left around this diagram is induced by the isotropy representation ρ. It follows that its composition with $\Delta(\theta)_*$ is the characteristic homomorphism as used in this paper (more precisely after further composition with the map γ_* of (7.19)). On the other hand the counter-clockwise map

$$\varepsilon^* \circ \text{jet}^*: H_c(\mathfrak{L}_0(\mathbf{R}^q), O(q)) \to H_c(\bar{\mathfrak{g}}, K)$$

(again composed with the map γ_* of (7.19)) is the characteristic homomorphism à la Bernstein-Rosenfeld and Bott-Haefliger. The fact that $\Delta(\alpha)_*$ is an isomorphism is due to Gelfand-Fuchs **[GF 1]**, **[GF 2]**. $\Delta(\alpha)$ also induces an isomorphism $H(W(\mathfrak{gl}(q))_q) \to H_c(\mathfrak{L}_0(\mathbf{R}^q))$.

Addendum. At the time of completion of this report, Fuchs announced the following remarkable results **[F 2]**, **[F 3]**. Consider the q-dimensional torus T and foliations on products $X \times T$ transverse to the projection $X \times T \to X$. These foliations are classified by homotopy classes of maps into the homotopy theoretic fiber $B\mathfrak{L}(T)$ of the canonical map B Diff $T^\delta \to B$ Diff T, where Diff T^δ denotes the group of diffeomorphisms of T with the discrete topology. In the previous notations $F = T$, $\bar{K} = T$ and $K = \{e\}$. The functoriality of γ_* in (7.19) defines by universality a map

$$\tilde{\gamma}_*: H_c(\mathfrak{L}(T)) \to H(B\mathfrak{L}(T)),$$

and Fuchs establishes the following result.

THEOREM **[F 2]**. *The composition*

$$\tilde{\gamma}_* \circ \text{jet}^*: H_c(\mathfrak{L}_0(\mathbf{R}^q)) \to H(B\mathfrak{L}(T))$$

is injective.

By universality there is a map $f: B\mathfrak{L}(T) \to F\Gamma_q$ into Haefliger's classifying space for foliations with trivial normal bundle, and a commutative diagram

$$\begin{array}{ccc} H_c(\mathfrak{L}_0(\mathbf{R}^q)) & \xrightarrow{\tilde{\gamma}_* \circ \text{jet}^*} & H(B\mathfrak{L}(T)) \\ \cong \Big\uparrow \Delta(\alpha)_* & & \Big\uparrow f^* \\ H(W(\mathfrak{gl}(q))_q) & \xrightarrow{\tilde{\Delta}_*} & H(F\Gamma_q, \mathbf{R}) \end{array}$$

COROLLARY **[F2]**. $\tilde{\Delta}_*$ *is injective.*

What still remains of interest in the earlier stated linear independence results is of course the fact that they already hold for single specific nonsingular homogeneous foliations on finite-dimensional manifolds. In fact it has been pointed out by Fuchs **[F 1]** that there are classes in $H_c(\mathfrak{L}(\mathbf{R}^q))$ which cannot be nontrivially realized for homogeneous foliations in the finite-dimensional context.

Fuchs similarly announces in **[F 2]** the injectivity of the relative universal characteristic homomorphism Δ_*. In **[F 3]** there is further an announcement of the result that all classes except those which are rigid by Heitsch's theorem **[HT 1]** are in fact variable.

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