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REPRESENTATION THEOREMS FOR MULTIFUNCTIONS AND ANALYTIC SETS

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We denote by (T, Σ) a measurable space and by X, Y, Z metrizable topological spaces. A *multifunction* from T into X is a mapping $t \longrightarrow M(t)$ which assigns a set $M(t) \subset X$ (possibly empty) to each $t \in T$. The set

$$\operatorname{dom} M = \{t \in T | M(t) \neq \varnothing\}$$

is called the *domain* of M. The multifunction M is called Σ*-measurable* if

$$M^{-1}(C) = \{t \in T | M(t) \cap C \neq \varnothing\}$$

belongs to Σ whenever $C \subset X$ is closed.

If (T, Σ), (T', Σ') are measurable spaces, then $\Sigma \otimes \Sigma'$ denotes the product σ-algebra generated by rectangles $E \times E'$, $E \in \Sigma$, $E' \in \Sigma'$. By $B(X)$ we denote the algebra of Borel subsets of X. A set $E \subset T$ is called Σ*-analytic* if it can be represented as a result of the A-operation of Suslin applied to elements of Σ. An equivalent definition: E is Σ-analytic if there is a Polish space X and a set $A \in \Sigma \otimes B(X)$ such that E is the projection of A on T (cf. [3]).

THEOREM 1. *Let X be a Polish* (*resp. compact metrizable*) *space, and let M be a Σ-measurable multifunction from T into X such that the sets $M(t)$ are closed. Then there exist a Polish* (*resp. compact metrizable*) *space Z and a mapping $f(t, z)$: $T \times Z \longrightarrow X$ such that*

(i) *f is continuous in z and Σ-measurable in t;*

(ii) *for all $t \in \operatorname{dom} M$, one has $M(t) = f(t, Z)$, the range of $f(t, \cdot)$.*

If X is a separable Fréchet space and all sets $M(t)$ are convex, then there is a pair (Z, f) satisfying (i), (ii) *and such that Z is a closed convex subset in another separable Fréchet space Y, and for all $t \in T$, the mapping $z \longrightarrow f(t, z)$ is the restriction on Z of a linear nonexpansive mapping from Y into X.*

Taking a dense countable set $\{z_1, \ldots\}$ in Z, one gets a dense countable family of measurable selectors for M. The existence of such a family was established by Castaing [2]. In our case, however, this family is rather "well arranged". In fact, the method we have used to prove the theorem demands explicit construction of such a family (with the help of the selection theorem of Rokhlin [8] and

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Kuratowski and Ryll-Nardzewski [6]), to define the space Z and the mapping f. The latter is carried out much in the same manner as in Ekeland's work [4], in which a similar result for continuous convex-and-compact-valued multifunctions was proved.

We shall say that the measurable space is *complete* if Σ contains all Σ-analytic subsets of T.

THEOREM 2. *Assume that (T, Σ) is a complete measurable space and that X is a Lusin space. Let $A \subset T \times X$ be $\Sigma \otimes B(X)$-analytic, and denote by E the projection of A on T. Then there are a Polish space Z and a mapping $f\colon E \times Z \longrightarrow X$ which is Σ-measurable in t, continuous in z and such that A is the range of the mapping $(t, z) \longrightarrow (t, f(t, z))$ from $E \times Z$ into $E \times X$.*

V. Levin has brought my attention to the fact that this theorem can be restated in a stronger form: it is possible to assume the measurable space complete but require instead that f be measurable with respect to the σ-algebra generated by Σ-analytic subsets of T, rather than with respect to Σ.

For any $z \in Z$, the mapping $t \longrightarrow f(t, z)$ is a measurable selector for A. Thus Theorem 2 implies a result on selectors which turns out more general than the corresponding results of Aumann [1] and Sainte-Beuve [9], because our assumptions on the measurable space are, in fact, weaker. (There is a complete measurable space whose algebra does not contain all universally measurable sets.) Note that selection theorems of such type (for analytic sets) go back to Lusin [7] and Yankov [10].

THEOREM 3. *Let X be a compact metrizable space, and S be a Banach space of continuous mappings from X into R^n such that the imbedding $i\colon S \longrightarrow C(X, R^n)$ is continuous. Assume also that either $i(S)$ is F_σ in $C(X, R^n)$ or the measurable space (T, Σ) is complete.*

Let there be given a multifunction $(t, x) \longrightarrow Q(t, x)$ from $T \times X$ into R^n such that

(a) *$Q(t, x)$ is nonempty convex and compact for each t, x;*

(b) *for all $x \in X$, the multifunction $t \longrightarrow Q(t, x)$ is measurable; for all $t \in T$, the multifunction $x \longrightarrow Q(t, x)$ is Hausdorff continuous;*

(c) *for all t, x and all $z \in Q(t, x)$, there is a mapping $h(\cdot) \in S$ such that $h(x) = z$ and $h(u) \in Q(t, u)$ for all $u \in X$.*

Then there are a Polish space V and a mapping $g\colon T \times X \times V \longrightarrow R^n$ such that

(i) *g is measurable in t and continuous in (x, v);*

(ii) *$g(t, \cdot, v) \in S$ for any t, v;*

(iii) *$Q(t, x) = g(t, x, V)$ for all t, x.*

It follows from the theorem that the differential inclusion $\dot{x} \in Q(t, x)$ can be rewritten as an ordinary differential equation with control $\dot{x} = g(t, x, v)$ in such a way that g will preserve regularity properties of Q. For instance g can be taken Lipschitz in x if $Q(t, \cdot)$ is Lipschitz, or C^1 in x if for every t the multifunction $Q(t, \cdot)$ admits a rich collection of C^1-selectors. The fact that such reduction is possible with g merely continuous in x was proved earlier by Ekeland and Valadier [5].

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