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Le mouvement brownien relativiste, by Jean-Pierre Caubet, Lecture Notes in Mathematics, no. 559, Springer-Verlag, Berlin, Heidelberg, New York, 1976, ix + 212 pp., $10.20.

It is just 25 years since Imre Fényes [2] discovered the Markov process associated with a solution of the Schrödinger equation. This process is easy to describe. Suppose that we have a solution ψ of the Schrödinger equation

$$\frac{\partial}{\partial t}\psi(x, t) = i\left(\frac{1}{2}\Delta - V\right)\psi(x, t). \tag{1}$$

Here x ranges over $\mathbf{R}^n$ and t over $\mathbf{R}$ and for each t the function $\psi_t(x) = \psi(x, t)$ is in $L^2(\mathbf{R}^n)$. We have set $\hbar = m = 1$, and V is the operator of multiplication by a possibly time-dependent real function (also denoted by V). The quantity $\|\psi_t\|_2^2$ is independent of t and may be normalized to 1, so that $\rho_t = |\psi_t|^2$ is the density of a probability measure on $\mathbf{R}^n$. We may write $\psi = \exp(R + iS)$ and set $u = \operatorname{grad} R$, $v = \operatorname{grad} S$, $b = u + v$. Then the diffusion process with drift b–that is, the Markov process satisfying

$$dx(t) = b(x(t), t)\,dt + dw(t) \tag{2}$$

where w is the Wiener process–and with initial distribution ρ_0 has the probability distribution ρ_t at all times t. Conversely, given such a diffusion process (2) we may let b_* be the backward drift and set $u = (b - b_*)/2$, $v = (b + b_*)/2$. Then u is the gradient of $\frac{1}{2}\log\rho$. If we assume that v is also a gradient then there is a unique V and a unique solution ψ of (1) such that with $\psi = \exp(R + iS)$ we have $u = \operatorname{grad} R$ and $v = \operatorname{grad} S$.

To get a better idea of these processes let us consider a few examples.

Consider the Wiener process itself where as customary we require the particle to be at the origin at time 0. Then the drift $b(x, t)$ is 0 for $t > 0$, but for $t < 0$ the particle is destined to go from x to the origin in $|t|$ units of time and the drift is $b(x, t) = x/t$. Similarly $b_*(x, t) = 0$ for $t < 0$ but $b_*(x, t) = x/t$ for $t > 0$. Now it is straightforward to compute u, v, R, S, ψ, and V. We find

$$\psi = (\pi t)^{-n/4}\exp\left(-\frac{1}{2}\frac{x^2}{|t|} + i\frac{1}{2}\frac{x^2}{t}\right),$$

which satisfies (1) with $V = x^2/4t^2$. The graph of this time-dependent potential is a paraboloid which is very shallow for large negative t but which snaps shut as $t \to 0$ and then opens out again, forcing the particle to go through the origin at $t = 0$. Thus this process is not free; it is subject to a force $F = -\operatorname{grad} V = -x/2t^2$.

A shortcut for computing V is available from the fact [3], [4] that Newton's law $F = ma$ holds, where a is the mean acceleration defined as follows. For a stochastic process x we define

$$Dx(t) = \lim_{h \to 0+} E\left\{ \frac{x(t+h) - x(t)}{h} \,|\mathcal{P}_t \right\},$$

$$D_* x(t) = \lim_{h \to 0+} E\left\{ \frac{x(t) - x(t-h)}{h} \,|\mathcal{F}_t \right\}$$

(where $\mathcal{P}_t$ is the past at time t and $\mathcal{F}_t$ is the future at time t), and the mean acceleration is $a = \frac{1}{2} DD_*x + \frac{1}{2} D_*Dx$. For the example of the preceding paragraph we readily compute that $a = -x/2t^2$, so that (since we have set $m = 1$) $F = -x/2t^2$ and $V = x^2/4t^2$.

For contrast, let us give an example of a free process (one subject to no forces, with mean acceleration zero). The complex Gaussian function

$$\psi_a(x, t) = \left(\pi \frac{a^2 + t^2}{4a}\right)^{-n/4} \exp\left(-\frac{x^2(a - it)}{a^2 + t^2}\right),$$

where a is a strictly positive constant, is a solution of the free Schrödinger equation ($V = 0$). We find that $b(x, t) = (t - a)x/(a^2 + t^2)$. For large values of t this is approximately x/t, the velocity needed to bring a particle from the vicinity of the origin to x in time t, and the process looks like the Wiener process with a normally distributed random drift term.

The Schrödinger equation is linear and therefore the principle of superposition holds. Consider the solution of the free Schrödinger equation given by

$$\psi(x, t) = \alpha(\psi_a(x - x_0, t) + \psi_a(x + x_0, t))$$

where α is a normalization constant. This describes the famous double slit through experiment where a beam of particles passes through two slits at x_0 and $-x_0$. (For simplicity the description is with respect to a Galilean frame of reference moving with the beam.) If a is small compared to x_0 then with large probability at time 0 the particle is either close to x_0 or close to $-x_0$, but for times t comparable to a we see that ρ_t is quite complicated. This complicated diffraction pattern is the probability distribution for a Markov process, a random motion in which the particle has no memory of the past. Yet if we did not know better we would be tempted to attribute this pattern to an interference effect for some sort of wave motion.

Consider finally the time-independent function $\psi(x, t) = \pi^{-1/2}e^{-|x|}$, where $n = 3$. This is the ground state of the hydrogen atom and satisfies (1) with $V = 1 - 2/|x|$. In quantum mechanics, once one has found the wave function there is no more to be said, but for the stochastic interpretation of the Schrödinger equation this is just the beginning. The wave function determines the diffusion process, and then we can ask what the trajectories of the process look like. In the present example we find that $b(x, t) = -x/|x|$. The particle diffuses with a constant tendency to head toward the origin (which is offset by the fact that there are more directions away from the origin to choose from than there are directions heading towards the origin). If one shows a movie of a particle performing this process then no one will be able to tell whether the movie is being run forward or backward. Real wave functions which are eigenstates of the hydrogen atom for higher energies have nodal surfaces which divide space into noncommunicating regions. The diffusing particle never crosses a nodal surface.

The stochastic interpretation of the Schrödinger equation is intriguing. It is at variance with the traditional interpretation of quantum theory because it presents a picture of continuous motion described in classical probabilistic terms.

The book under review starts off (Chapters 2–4) with some standard material in probability theory, starting at the beginning and moving rapidly to the theory of diffusion on a Riemannian manifold. Then Caubet treats the diffusion processes associated with solutions of the general Schrödinger equation

$$\left(i\frac{\partial}{\partial t} - V\right)\psi = \frac{1}{2}\left(\frac{1}{i}\nabla - A\right)^2\psi$$

on a Riemannian manifold. A beautiful and important application of the extra generality given by a Riemannian manifold is obtained from the manifold $\mathbf{R}^3 \times SU(2)$. Dankel [1] used this to obtain the stochastic interpretation of the Pauli equation, and thus to give a nonrelativistic stochastic treatment of spin. Caubet (apparently unaware of Dankel's work) presents an account of this topic.

The reviewer has trouble understanding the author's intentions beginning with §5.4 entitled "Mouvement brownien relativiste". The author continues to use the language of diffusion theory but replaces the Laplacean on Euclidean space by the d'Alembertian on Minkowski spaces. It does not seem that stochastic processes having the desired properties exist. It may be that the author intends diffusion theory to serve merely as a suggestive analogy when he comes to the relativistic theory because he says in the Introduction (p. 17): "Il y a donc une différence essentielle entre le processus de Wiener et le mouvement brownien relativiste: dans le premier la trajectoire est décrite par la particule ponctuelle tandis que dans le second la trajectoire est au niveau quantique sans réalité physique, la particule à ce niveau étant constituée de l'onde elle-même (ou d'une partie de l'onde)."

The stochastic interpretation of the Schrödinger equation offers many challenging problems to probabilists. One problem is to characterize unaccelerated diffusions. How can one tell by observing the trajectories of a diffusion process whether the potential V in the corresponding Schrödinger equation is zero?

Suppose we have a Markov process corresponding to a solution of the free Schrödinger equation. What is the asymptotic behavior of the trajectories? Does $x(t)/t$ tend to a limit with probability one as $t \to \infty$? If so, what is the correlation of the limits as $t \to \infty$ and $t \to -\infty$? The same question can be asked for solutions belonging to the continuous spectrum for Schrödinger equations with potential V converging to zero sufficiently fast at infinity. How does one distinguish probabilistically between solutions belonging to the discrete and continuous portions of the spectrum?

Is there a probabilistic interpretation of the superposition principle? What does it mean for one Markov process to be a superposition of several others?

The stochastic interpretation applies equally well to Schrödinger equations describing an ensemble of k particles. Suppose that we are not allowed to observe trajectories directly, but only to observe the position of k particles at one fixed time. (Then we know that the predictions of the stochastic interpretation agree with the predictions of quantum mechanics.) We are free to impose any time-dependent potentials we wish and to consider $k - 1$ of the particles as observing instruments. How much information can we obtain about the trajectory of the remaining particle in this way?

The stochastic interpretation gives a clear meaning to the notion of the probability that a particle (in a process corresponding to a solution of the Schrödinger equation) is ever in a given region during a given interval of time. The orthodox theory of quantum mechanical measurement is restricted to observations made at one fixed time. Is there a quantum mechanical definition of this probability which agrees with the probability given by the stochastic interpretation?

There remains the problem of developing a stochastic relativistic theory. Theories of relativistic interaction appear to require fields. In recent years probabilistic techniques have played a large role in constructive quantum field theory, but the random fields have been constructed on Euclidean space, rather than Minkowski space, and the results for quantum fields have been obtained by analytic continuation. This is analogous to studying the Schrödinger equation by means of the corresponding heat equation, and then analytically continuing in time. The field-theoretic analogue of the stochastic interpretation of the Schrödinger equation remains to be constructed.

ADDED IN PROOF. Some of the questions raised here have been answered by David Shucker in a Princeton thesis (to appear).

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Grundzüge der universellen Algebra, by Herbert Lugowski, Teubner-Texte zur Mathematik, B. G. Teubner Verlagsgesellschaft, Leipzig, 1976, 238 pp., DM 19.50.

Universal algebra, as a method, has been extremely fruitful; by contrast, as an independent discipline it appears a little arid, owing to the fact that so many of its results have been somewhat less universal in their application. Perhaps the subject has developed best when working in harness with another part of mathematics, such as logic or category theory, and this is reflected in more recent books such as [1], [2]. Another field which would provide a good