groups is summarized . . . ". From a logical point of view, then, the book starts from scratch. In actual fact, however, you should either have some familiarity with algebraic numbers, cohomology and locally compact groups, or be in the presence of someone who does. If not, start with Lang. Otherwise there is a chance that you will get bogged down in the machinery. On the other hand, if you are ready for the book, and if you master it, you will have a complete understanding of class field theory in the modern medium and will be ready to approach difficult and active areas of research like the arithmetic theory of algebraic groups, modern analytic number theory, and nonabelian class field theory. If you are looking for a cohomological development of class field theory in introductory book form, then the only feasible alternative to Iyanaga is Cassels-Fröhlich. There everything is done in 203 pp. Iyanaga takes 400. Some readers will find Cassels-Fröhlich sketchy, others will find Iyanaga ponderous. While the definitive text in the modern medium remains to be written, and writing it will require enormous effort, even a touch of genius, the authors of *The theory of numbers* are to be thanked and congratulated for successfully completing a big task and for enriching the literature with a coherent account of class field theory in the modern spirit. Needless to say, *The theory of numbers* should be in the possession of anyone interested in algebraic number theory.

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Stability of fluid motions. I, II, by Daniel D. Joseph, Springer Tracts in Natural Philosophy, vol. 28, Springer-Verlag, New York, 1976, xiii + 282 pp., $39.80, and xiv + 274 pp., $39.80.

The distinction between laminar flow and turbulent flow of a fluid is, in the first place, a matter of everyday experience. Broadly speaking, laminar flow is regular and smooth, while turbulent flow is characterized by the irregularity and random nature of the motion. Although the division between these two types of flow is not always sharp, and although a precise definition of turbulence is difficult to formulate, there is sufficient experimental evidence to indicate that the classification of fluid motions into two states, laminar and turbulent, is a very good approximation to real behaviour, at least in so far as common fluids under normal conditions are concerned.

The serious scientific investigation of turbulence begins with the experiments of Reynolds (1883). Apart from many incisive observations concerning the nature and effects of turbulent flow in pipes, Reynolds remarks on the question of the transition from laminar flow to turbulent flow. He speaks, in fact, of the "change" from one state to the other, with the important implication that the two states cannot, in general, both occur under a given set of conditions. When the conditions are varied, transition from one flow state to another may occur; Reynolds suggests, moreover, that the variation of conditions can be measured in terms of a single dimensionless parameter which combines the speed of the flow, the density and viscosity of the fluid and the diameter of the pipe. This parameter is now called the Reynolds number.

In the same classic paper Reynolds poses the question of why and how transition from laminar flow to turbulent flow takes place, and he proposes an answer, which he attributes to Stokes, in terms of the breakdown of stability of the laminar flow. "The general cause of the change from steady [i.e. laminar] to eddying [i.e. turbulent] motion was in 1843 pointed out by Professor Stokes, as being that under certain circumstances the steady motion becomes unstable, so that an indefinitely small disturbance may lead to a change to sinuous motion." Reynolds postulates that the circumstances referred to are that the dimensionless parameter, gradually increased, reaches a certain critical value at which the laminar flow becomes unstable to infinitesimal disturbances.

Although it is now known that Reynold's postulate is too simplistic to explain the onset of turbulence, it has nevertheless become universally accepted that the study of transition from laminar flow to turbulent flow must begin with a study of the stability of the former. Since the days of Reynolds the understanding of the transition phenomenon has been one of the major objectives of fluid-mechanical research, and in this activity the analysis of the stability of laminar flows has played a central role.

The formulation of the problem in mathematical terms is not at all difficult. It is not in dispute that the motion of a Newtonian fluid (either laminar or turbulent) is described by the Navier-Stokes equations of momentum conservation and the continuity (mass-conservation) equation. For an incompressible, viscous fluid these equations are, respectively,

$$\partial U/\partial t + (U \cdot \nabla)U = -\nabla P + (1/R)\nabla^2 U \tag{1}$$

and

$$\nabla \cdot U = 0, \tag{2}$$

where U is the velocity vector, P the pressure, R the Reynolds number, and where t represents time. Equations (1) and (2) are sufficient to describe simple flow situations, but have to be extended when other effects, such as variations in density, a magnetic field or heat sources, are present.

To specify the mathematical problem one generally assumes that the fluid is confined within a bounded domain V, on whose boundary ∂V the velocity vector is required to satisfy prescribed conditions. The basic laminar-flow state, whose stability is to be examined, is then taken to be represented by a solution $\bar{U}(x)$, $\bar{P}(x)$ of equations (1) and (2) which satisfies the boundary conditions. Usually, though not necessarily, this solution is supposed to depend only on the spatial position vector x and not on the time t.

A stability analysis of the basic state is achieved by setting

$$U = \bar{U} + u(x, t), \quad P = \bar{P} + p(x, t) \tag{3}$$

in (1)–(2) and the boundary conditions. Assuming that the disturbance is "switched on" at time $t = 0$, one is then led to the following initial-boundary-value problem: to solve in V the equations

$$\frac{\partial u}{\partial t} + (U \cdot \nabla)u + (u \cdot \nabla)U + (u \cdot \nabla)u = -\nabla p + \frac{1}{R}\nabla^2 u, \tag{4}$$

$$\nabla \cdot u = 0, \tag{5}$$

with

$$u = 0 \quad \text{on } \partial V \tag{6}$$

and

$$u(x, 0) = u_0(x) \quad \text{in } V. \tag{7}$$

The stability of the basic state is determined from the behaviour of the solutions (4)–(7) for arbitrary initial disturbance $u_0(x)$. More precisely, define

$$E(t) = \tfrac{1}{2}\langle |u|^2 \rangle, \quad E(0) = \tfrac{1}{2}\langle |u_0|^2 \rangle, \tag{8}$$

where $\langle \cdot \rangle$ denotes average with respect to x over V. Then the basic state is called *stable* (more accurately, asymptotically stable in the mean) if $E(t) \to 0$ as $t \to \infty$ for all $E(0)$, *unstable* if $E(t) \to \infty$ as $t \to \infty$ for some $E(0)$, and *conditionally stable* if there is a $\delta > 0$ such that $E(t) \to 0$ as $t \to \infty$ when $E(0) \leqslant \delta$.

Although the problem is easy to formulate, to determine its solution is altogether another matter. The major source of difficulty lies in the nonlinear character of the equations (4), and it is therefore natural to attempt to make progress by *linearizing* equations (4) through the deletion of the term $(u \cdot \nabla)u$. The linearized stability problem admits solutions of the form

$$e^{\sigma_i t}\phi_i(x), \qquad i = 1, 2, \ldots, \qquad \sigma_i = \sigma_i(R), \tag{9}$$

and from these a conditional stability result can be obtained: the basic laminar state is stable to sufficiently small initial disturbances if $\mathrm{Re}\,\sigma_i < 0$ for all i, and is unstable to an infinitesimal disturbance if $\mathrm{Re}\,\sigma_i > 0$ for some i. This important extension of the well-known principle of linearized stability for ordinary differential equations was achieved by Prodi (1962), Kirchgässner and Sorger (1968) and Sattinger (1970).

The linearized stability problem is a parameter-dependent, linear eigenvalue problem. It is required to determine the exponents σ_i as functions of R and, especially, the critical value R_c of R at which the real part of an exponent σ_i first changes from negative to positive as R increases.

Determination of critical values of R (or of other parameters where appropriate) was the main preoccupation of research workers in the field during the period 1920–1960. By the end of this period it could be said that a satisfactory resolution of the linearized stability problem had been obtained.

In certain particular cases the problems involved were solved by and stimulated the development of powerful techniques of asymptotic analysis.

The unravelling of the linearized stability problem was a necessary first step in the elucidation of the transition process, but it was soon abundantly clear that many important questions remained unanswered. As a supposed explanation of transition, linearized stability theory has some serious deficiencies, of which the following are perhaps the most crucial.

1. The attainment of the critical value of R is a *sufficient* condition for the onset of instability, but since the linear theory gives a conditional stability result for infinitesimally small disturbances it cannot be expected to provide a *necessary* condition for the onset of instability.

2. It is known from experiments that some flows become unstable at a value of R substantially less than the value R_c of linear theory. Presumably such flows are stable to infinitesimal disturbances and unstable to some finite disturbances. Linearized stability theory cannot predict the onset of instability in such cases.

3. To make matters worse, there are certain flows which have no finite critical value of R, even though they are known to change from laminar to turbulent at some finite R. Evidently linear theory has no bearing on the transition process in these cases.

4. As equation (9) shows, the linear theory predicts that an unstable disturbance grows exponentially without limit. Exponential growth may be an adequate description of the behaviour of a disturbance in its incipient stage, but it cannot be acceptable as a description of the evolution of the disturbance over a period of time.

Over the past two decades research endeavour in the transition problem has been concentrated on attempting to resolve the questions listed above. Mathematically this has meant a return to the study of the nonlinear problem defined by equations (4)–(7), and the formidable nature of this system has necessitated special modes of attack on particular aspects.

One approach which has met with considerable success has been concerned with the determination of a *sufficient condition for stability*. Is there a value R_E, say, of R such that the basic laminar flow is unconditionally stable when $R < R_E$? A first attempt to answer this question was made as early as 1907 by W. Orr, and subsequent contributions were made by several authors. The modern approach, however, is due to Serrin (1959), whose pioneering work stimulated and laid the basis for important results by others, especially D. D. Joseph.

The computation of a sufficient condition for stability has been called *energy theory*, and it takes its name from the fact that it begins with the equation

$$dE/dt = -\langle u \cdot \nabla U \cdot u \rangle - (1/R)\langle |\nabla u|^2 \rangle \tag{10}$$

for the rate of change of the energy of disturbances. (10) is easily derived by taking the scalar product of (4) with u, and then effecting the spatial average associated with (8). Using Poincaré's inequality one obtains from (10) the relation

$$dE/dt \leqslant \{-\langle u \cdot \nabla U \cdot u \rangle / \langle |\nabla u| \rangle^2 - 1/R\}E. \tag{11}$$

If now a number R_E is defined by

(12) $$1/R_E = \max_{u \in H} -\langle u \cdot \nabla U \cdot u \rangle / \langle |\nabla u|^2 \rangle,$$

where H is a set of sufficiently-smooth, divergence-free vector functions in V, then (12) becomes

(13) $$dE/dt \leqslant (1/R_E - 1/R)E.$$

It follows that $E \to 0$ as $t \to \infty$ when $R < R_E$, which is sufficient for stability to all disturbances belonging to the set H. The calculation of R_E is a relatively easy problem of calculus of variations.

There are a few configurations in which $R_E = R_C$; in these cases this value of the parameter R represents an unconditional boundary between stability and instability of the basic laminar flow. In most flow situations, however, $R_E < R_C$ and the boundary lies somewhere between the two values; in such cases the location of the unconditional stability boundary is not known. When the difference between R_C and R_E is great, the energy-theory result is clearly very conservative and its usefulness is consequently diminished.

Another approach to the nonlinear stability problem concerns itself with the evolution of unstable disturbances. It has been pointed out earlier that the exponential growth predicted by linear theory cannot be expected to persist beyond an early stage. A mechanism whereby nonlinear effects could come into play was proposed by Landau (1944). If solutions of the linearized stability problem were written in the form

(14) $$A_i(t)\phi_i(x)$$

rather than in the form (9), then the amplitude function $A_i(t)$ could be a solution of the equation

(15) $$dAi/dt = \sigma_i A_i + \beta_i A_i^3 + \cdots.$$

In this equation σ_i is the linear stability exponent, and therefore when nonlinear terms are neglected (14) and (15) combine to give (9). If, on the other hand, nonlinear terms are not ignored, solutions of (15) could evolve to new steady states at $t \to \infty$; these might represent solutions of the flow equations other than the basic laminar flow.

Whether or not evolution to a new solution occurs depends on the constant β_i in (15). The computation of this constant is a somewhat laborious matter, but has been achieved successfully for many different flow problems. Landau's equation (15) is appropriate only when loss of stability is due to the change of sign of a real eigenvalue; when the sign of the real part of a complex eigenvalue changes, a modified version of this equation is required.

The fact that explicit results could be obtained from an equation of Landau type was demonstrated forcefully by Stuart (1960), Palm (1960) and many others, subsequently. On the other hand, this approach has been criticized on the grounds that it is purely heuristic, that it lacks complete mathematical justification and, perhaps most important, that it does not provide a global theoretical framework within which the transition problem can be viewed. Although these criticisms are all largely valid and although a more rigorous theory has evolved over the past few years, there is no doubt that the work of

Stuart, Palm, Busse and others in the 1960's contributed a great deal to the elucidation of the stability problem.

The concept that a laminar flow, on losing stability, is replaced immediately by a turbulent flow does not usually accord with observation. On the contrary, experiments suggest that there are frequently occasions when a laminar flow is replaced by another, more complicated, possibly time-periodic, laminar flow. As R increases still further, this new flow may become unstable and be replaced by yet another laminar flow of an even more complicated structure. The really important contribution of Landau (1944) was the conjecture that transition to turbulence is a process of repeated loss of stability and repeated bifurcation from a laminar state to a more complex laminar state. Eventually, presumably, the complexity is such that the state which is attained may reasonably be called turbulence.

Landau's conjecture is not nowadays regarded as providing a satisfactory description of the transition process. Nevertheless, the notion of bifurcation from one state to another, consequent on the loss of stability of the former, is the cornerstone of the modern theory of transition in fluids. Although *bifurcation theory* does not in itself answer all questions, it constitutes an impressive mathematical framework which incorporates both linear theory and the Landau-equation theory, and which potentially allows a much more extensive analysis of the transition problem than has been possible hitherto.

The book by Joseph under review, a single entity even though published in two parts, does not concern itself explicitly with either linear stability theory or the heuristic nonlinear theory based on the Landau amplitude equation. It concentrates instead on an exposition of bifurcation theory for fluid stability, and includes an ample discussion of energy theory. To some extent, moreover, it is a research monograph; the author has made outstanding contributions to both energy theory and bifurcation theory, and much of the book is written around these contributions.

In Chapters 1 and 2 the author provides an excellent introduction to the subject. Chapter 1 contains an exposition of the determining equations of fluid motion and a statement of the stability problem; the latter is accompanied by a discussion of the various stability criteria which can be applied and of their significance. In this chapter, too, there is a brief but lucid exposition of energy theory and its relation to the question of uniqueness of solutions of the equations of motion.

The key to the book, and the key also to bifurcation theory within the transition problem, is to be found in Chapter 2. The essential features of bifurcation theory as they relate to the system (4)–(7) may be summarized as follows. Note first that as a result of the transformation (3) the basic laminar flow appears as the null solution in (4)–(7). Suppose also that this null solution loses stability according to linear theory as R increases through R_C. Then, in the main, one or another of two possibilities can occur.

A. The loss of stability is as a result of the change of sign of one real eigenvalue σ as R increases through R_C. In this case a nontrivial, *time-independent* bifurcating solution emerges from the null solution at $R = R_C$. Depending on the structure of the particular problem this bifurcating solution may exist only when $R < R_C$ (*subcritical*) or only when $R > R_C$ (*supercriti-*

cal) or when both $R < R_C$ and $R > R_C$ (*two-sided*). Whatever the case may be, a subcritical bifurcating solution is unstable according to linear theory, and a supercritical bifurcating solution is stable.

B. The loss of stability is as a result of the change of sign of the real part of a complex eigenvalue as R increases through R_C. Then a nontrivial *periodic* solution bifurcates from the null solution at $R = R_C$. This bifurcating solution is necessarily *one-sided*, that is, it exists only when $R < R_C$ or only when $R > R_C$. Moreover, if it is subcritical it is unstable, while if it is supercritical it is stable.

These bifurcation theorems were obtained by Joseph (1971), Sattinger (1971) and Joseph and Sattinger (1972), and independently by Yudovich (1971) and Iooss (1972). The theorems generalize to the equations of fluid mechanics analogous results on bifurcation and stability for systems of ordinary differential equations obtained by Hopf (1942). An important feature of these bifurcation theorems is that they apply generally to the system (4)–(7), and consequently incorporate all specific results obtained for particular problems by other methods, such as the Landau equation.

For the most part the remaining five Chapters of Part I and the seven Chapters of Part II are concerned with demonstrating the application of bifurcation theory (and, to a lesser extent, energy theory) to particular flow situations. Chapters 3 and 4 are devoted to the problem of flow down a pipe of annular cross-section. Chapter 3 presents the results of energy theory for this problem; insofar as they determine a sufficient condition for stability, these results are quite complete, but also quite conservative. Chapter 4 is concerned with the application of bifurcation theory. This is one of the most difficult and most interesting problems, since here the bifurcating solution is periodic and subcritical. Because it is subcritical it is unstable, and this suggests the possible existence of another, stable branch of solutions at a higher amplitude. This notion is supported by the observational fact that disturbances grow at values of R well below R_C, but presumably such disturbances must initially be large enough to fall outside the conditional limit of linear theory. A mathematical description of such a mechanism has not yet been achieved.

The stability of Couette flow between rotating cylinders, to which Chapter 5 is devoted, is itself the subject of a considerable literature. The author presents a clear and balanced summary of the state of knowledge with regard to this problem.

Chapter 6 is concerned with the stability of a fluid driven down the annular region between cylinders which are also rotating and sliding; Chapter 7 treats the flow between concentric rotating spheres. Both these problems are mathematically extremely complicated and on occasion the underlying ideas tend to become submerged in the unavoidable complexity of calculations.

Part II commences with four Chapters, 8–11, related to various aspects of the so-called thermal convection problem. This is primarily concerned with the loss of stability associated with the heating of a fluid and the consequent variations in its density. In this situation the bifurcating solutions are usually time-independent and usually, though not always, one-sided and supercritical. (Chapter 10 studies circumstances in which the bifurcation is two-sided.) The supercritical bifurcating solutions are stable, but as the appropriate parameter increases they may lose their stability to new bifurcating solutions. This possibility, which is not yet fully resolved, is discussed in detail in Chapter 11.

The remaining three chapters of the book are concerned with special topics. Chapter 12 treats the variational theory of turbulence, in which certain properties of statistically stationary, possibly turbulent, flow are derived. Chapter 13 presents an account of recent advances, mainly due to the author, in the stability of viscoelastic fluids. In general the understanding of this subject is still in its infancy. Chapter 14 deals with the intriguing topic of the stability of interfaces between different fluids. There are, finally, several appendices concerned with certain mathematical questions which arise in the main text.

Joseph's book is a well-written and carefully argued account of the current state of stability theory for fluid motions. It is comprehensive in its treatment, albeit the emphasis reflects the author's interests and contributions. The reader will quickly become aware that, although the transition problem is not yet solved, great progress has been made during the past decade and the possibility of decisive further advances has been uncovered. The fact that many researchers are enticed by the fascinating and difficult mathematical questions involved in the transition problem is due in no small measure to the author's contributions. This book will undoubtedly serve to stimulate more exciting mathematical activity in this area.

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