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Differential equations and their applications, by M. Braun, Applied Mathematical Sciences, vol. 15, Springer-Verlag, New York, Heidelberg, Berlin, 1975, xiv + 718 pp., $14.80.

Applied mathematics cannot reasonably be described as a single field. Unlike pure mathematics, which does possess a unity and a definite historical tradition, applied mathematics today is a collection of subjects bound loosely together by their common reliance on mathematical notation, ideas and methods. Because of this, a project of writing an introduction to applied mathematics for undergraduates may seem to be a hopeless task.

Martin Braun has shown that one should not give up so easily. Certainly many people who think of themselves as applied mathematicians will not find their favorite circle of ideas in an elementary text on differential equations. What is to be found, besides an excellent leisurely development of differential equations, is an introduction to the interaction between mathematics and its applications.

Mathematics can play an important, sometimes crucial, role in the structure of other disciplines. The fundamental ideas and relations of a subject can sometimes be expressed quantitatively and unambiguously using mathematical notation. When this is done, a mathematical model of some aspect of the subject in question results. Mathematics then provides frameworks in which the relations in the model may be analysed and manipulated to yield predictions. These predictions may be compared with data gathered in the field to foster confidence in certain aspects of the model and to discover shortcomings of the model. This process can, and often does, lead to an interaction between mathematics and the discipline under study whereby the model is successively improved. Perhaps the most exciting aspect of the modeling process is when new phenomena come to light whose existence was not previously recognized. Of course mathematics as a pure subject can benefit from this interaction as well.

These aspects of the modeling process find expression in Braun's book by way of a sequence of case studies of various applications. This is certainly not a new idea, even in the context of elementary differential equations. Engineering students have been subjected to 'problems analysis' courses for many years. Such courses typically exploit the case method to teach model building, and, on the side, offer a swashbuckling approach to the elements of differential equations. As we shall see, the present text offers more than just an up-to-date version of such courses.

The text is aimed at students with a good first-year calculus sequence in hand. The pace is not intensive, with plenty of chat, in the best sense of the word. The extended examples are nicely woven into the fabric of a first course in the mathematical theory of differential equations.

The theory is developed carefully, with attention to detail and rigor, and is standard fare mathematically except that parts of Chapter four (stability theory carried as far as the Poincaré-Bendixson theorem) would not usually appear in a text at this level. Such systematic development of the purely mathematical ideas and the depth to which the mathematics is carried is not at all common in courses aimed at engineering and science majors. On the other hand, what sets Braun's book apart from most second and third year mathematics texts on differential equations is the large collection of marvelous applications of the theory, many taken directly from original research papers in other areas. These applications are developed at length. The student is exposed to the modeling process in its full range–first tentative try, analysis of the result and comparison with the underlying situation being modeled, and consequent refinements of the model. He sees clearly the power of some of the deeper mathematical ideas and he sees interesting mathematical results suggested by the application in question. This is a composite view that is too seldom encountered by both mathematics majors and majors in the other sciences and in engineering.

The applications in Chapters 1–3 include typical topics from mechanics, circuit theory, radiative decay and population growth. Even these fairly standard applications are given interesting settings. Here is one which uses only the very elementary theory. It is a nice recent addition to the arsenal of radiative dating techniques.

The usual law for natural radiative decay is

$$\frac{dN}{dt} = -\lambda N,$$

where $N(t)$ denotes the number of atoms of a decaying substance present at time t and λ is the decay constant for the substance in question. Note that λ has the units of inverse time and that $\log 2/\lambda$ is the half-life of the substance. The method to be explained presently was used to settle the authenticity of a painting attributed to Vermeer. (The story of the furor over this painting is quite interesting in its own right!)

White lead (half-life 22 years) is an important pigment which has been used by painters for centuries. It is typically mixed with uranium and a radium descendent (half-life 1600 years). In its natural state, the amount of white lead decaying to lead is replaced more or less exactly by the radium decaying to white lead. This steady state is interrupted by the chemical process that extracts the white lead. Following extraction, the amount of white lead is reduced by its more rapid decay rate until it comes into equilibrium with the much smaller amount of radium remaining.

Let $y(t)$ be the number of grams of white lead per gram of ordinary lead at time t, y_0 the amount of white lead per gram of ordinary lead at time t_0 when the white lead was extracted from the natural ore. Further, let $r(t)$ be the number of disintegrations of radium per minute per gram of ordinary lead at time t. If λ denotes the decay constant for white lead, then y is governed by the simple balance

$$\frac{dy}{dt} = -\lambda y + r(t), \qquad y(t_0) = y_0.$$

As interest is focused only on a span of about three centuries, it's a fair assumption that the amount of radium remains constant, and hence that $r(t) \equiv r$ is a constant. Then it follows easily that

$$y(t) = y_0 e^{-\lambda(t-t_0)} + \frac{r}{\lambda}\left(1 - e^{-\lambda(t-t_0)}\right).$$

Now $y(t)$ and r are easily measured quantities associated to the white lead of a particular painting. So if y_0 were known, then t_0 could be computed. Of course, y_0 is not known, and moreover the number λy_0 (the number of disintegrations of white lead per minute per gram of ordinary lead) is seen to vary over a wide range (roughly 10^{-1} to 10^2) in nature.

An exact dating of a painting, or at least of the manufacture of the pigment, is therefore out of the question. However, it is possible to distinguish a twentieth century forgery and a painting executed three centuries ago. For supposing that the painting is genuine, then $t - t_0$ is 300 years. Hence, upon measuring $y(t)$ and r, the number λy_0 may be determined. For the painting entitled "Disciples at Emmaus", which was authenticated by a leading art historian and purchased by the Rembrandt society for $170,000, the value of λy_0 turns out to be about 10^5. This is unacceptably large and shows that the hypothesis that the painting is three hundred years old must be false.

This sample is not unique in its interest and originality. The problem of determining the velocity of an object dropping under gravity through water is solved and used to examine a disposal method for radioactive waste. The resonance experienced by a mechanical system forced at its natural frequency is used to give a qualitative explanation of the famous Tacoma Narrows bridge disaster. A more unusual application of the very elementary theory is the analysis of a model used in the detection of diabetes.

The examples from Chapter four are even more impressive. A model for predicting arms production by nations is developed and analyzed using linear stability theory. This yields criteria for predicting when an international political situation is likely to lead to war. Some of the deeper applications, using the nonlinear stability theory, are to prove the competitive exclusion principle in biology (two different species cannot earn their living in exactly the same way) and discover the asymptotic states of epidemics. In these latter applications, the going is decidedly nontrivial and the reader gains some appreciation of how insights and requirements developed from the underlying phenomena being modeled can be very useful in more challenging mathematical situations.

An instance, worth relating here, is based on the elegant analysis, given by Volterra, of a problem posed by the Italian biologist D'Ancona. In the mid 1920's, D'Ancona came across an interesting fact in the course of his research. It seemed that of the various types of fish caught in the Mediterranean and brought into various Italian ports, the percentage of selachins (sharks, skates, rays, etc.) rose dramatically during World War I. He reasoned that this was due to the decreased intensity of fishing during the period of the war. He could not account for this phenomenon biologically. and so he posed the problem to Volterra.

Volterra commenced by separating fish into the prey population $x(t)$ and the predator population $y(t)$. He continued by postulating that the food fish (the prey) don't compete very intensively among themselves for food since their food supply is very abundant and the fish population is not very dense. With no predators around, the prey population would therefore grow according to the simple Malthusian law, namely

$$\frac{dx}{dt} = ax(t)$$

where a is a positive constant.

This rule plainly needs modification in the presence of the predators, and Volterra presumed the number of contacts of predators and prey per unit time to be proportional to the product $x(t)y(t)$. Hence he arrived at the modified equation,

$$\frac{dx}{dt} = ax(t) - bx(t)y(t),$$

governing the growth of the prey population. Here both a and b are positive constants at this level of approximation.

For the predators, however, there is a natural rate of decrease, proportional to their present numbers, which is offset by the availability of food. Thus Volterra arrived at the equation,

$$\frac{dy}{dt} = -cy(t) + dx(t)y(t),$$

where c and d are positive constants.

This system of equations may be analyzed and the following conclusions derived. First, any solution that begins in the positive quadrant of the x, y plane remains there subsequently. (Hence we don't have to worry about negative numbers of fish!) Secondly, all solutions $x(t)$, $y(t)$ that have their initial value, $x(0), y(0)$ say, both positive are necessarily *periodic* functions.

Now the data of D'Ancona was based on average catches over a period of one year. Hence in order to compare the solutions of Volterra's system with the data, temporal averages of the solutions of Volterra's system need to be computed. It is a nice example where the requirements of the modeling situation can lead to an interesting mathematical result. For if $x(t), y(t)$ is a positive quadrant solution of period T, then the average values,

$$\bar{x} = \frac{1}{T}\int_0^T x(t)dt \quad \text{and} \quad \bar{y} = \frac{1}{T}\int_0^T y(t)dt,$$

can be computed exactly. For note that $\dot{x}/x = a - by$, so by periodicity,

$$0 = \frac{1}{T}[\log x(T) - \log x(0)] = \frac{1}{T}\int_0^T \frac{\dot{x}(t)}{x(t)}\,dt$$

$$= \frac{1}{T}\int_0^T [a - by(t)]dt = a - b\bar{y}.$$

Solving, there appears $\bar{y} = a/b$. Similarly, $\bar{x} = c/d$.

So far the effects of fishing have been neglected. Suppose that fishing decreases the population of fish (all kinds) at a rate $\varepsilon > 0$ proportional to the population. The constant ε reflects how intensely the waters are being fished. With fishing included, the governing model becomes,

$$\frac{dx}{dt} = (a - \varepsilon)x(t) - bx(t)y(t), \qquad \frac{dy}{dt} = -(c + \varepsilon)y(t) + dx(t)y(t).$$

Provided $a - \varepsilon > 0$, this system is qualitatively the same as the original system that ignored fishing. Hence the average values for the system, modified to include fishing, are

$$\bar{x} = \frac{c + \varepsilon}{d} \quad \text{and} \quad \bar{y} = \frac{a - \varepsilon}{b}.$$

Plainly if ε is decreased from a given level $\varepsilon_0 < a$, the average population of food fish decreases and the average population of predators increases and thus we have a qualitative explanation of D'Ancona's data.

This result, known as Volterra's principle, has other applications. An important one is the prediction that, under certain conditions, insecticide treatments that kill both the pest and its enemies will actually increase the population of pests!

Braun goes on to raise criticisms of this basic two species model and to suggest and analyze modifications of the model to deal with, for example, more intensive competition within each species.

In addition to the lucid treatment of the theory, and the marvellous examples, there are a few other aspects of the book worth particular note. One is the collection of problem sets which are as rich and varied as the text itself. Others include a decent introduction to linear algebra and to numerical approximation of solutions of differential equations, including some rigorous error estimates.

The length of the book (more than 700 pages, though admittedly these are reproduced from typescript) could be a drawback. Students will probably find the book attractive enough to offset any negative reactions generated by its considerable bulk. But in a one quarter, or one semester length course, the instructor would certainly have to skip around in the text, and some of the continuity would thereby be lost unless the students could be persuaded to do a lot of independent reading.

Chapter 5 seems significantly weaker than the other four chapters. Students who have survived Chapter four can certainly be expected to handle a more substantial introduction to partial differential equations than just separation of variables combined with some simple applications of Fourier series. Moreover the chapter is bereft of exciting examples, and only the author's nice informal style can set it apart from any number of pedestrian introductions to the theory of partial differential equations. Considering the other accomplishments in the book, it is hard to be too damning on the points raised in the last two paragraphs.

In sum, it appears that Braun has successfully crossed two very different approaches to introductory differential equations and has thereby made a significant contribution to the teaching materials available at this level.

There is a more subtle way in which the book under discussion may play a useful role. To elucidate this aspect requires a few remarks that may at first seem to have little relevance to Braun's book on differential equations.

It is probably fair to say that in the nineteenth century no significant distinction was drawn between pure and applied mathematics. This was to change in the twentieth century, perhaps due in part to the higher standards of rigor and the increasing abstraction of pure mathematics. In response to the opening sentence in this review, one might reply that there was a time, not so long ago, when applied mathematics had a definite subject matter and a tradition to go along with it. This was exemplified clearly in Great Britain under the leadership of Jeffreys and G. I. Taylor for example. The subject matter of applied mathematics during this era might loosely be characterized as theoretical physics and mechanics. The tradition in mechanics was surely influenced strongly by Taylor, whose abilities at discovering phenomena and understanding their essential causes, building simple mathematical models and doing some elementary analysis to yield predictions which he then tested in the laboratory, were probably unparalleled in the history of science.

Applied mathematics saw an enormous growth during and after World War II and a number of centers sprang up, in government laboratories and in various universities, both in Britain and in the U. S. A. By now many of the universities around Britain have groups firmly established in this tradition and there are a number of strong and active enclaves in North America and elsewhere around the world. This 'classical' style of applied mathematics emphasized using a strong physical intuition in conjunction with mathematical formalism. In the hands of its able practitioners, this mix of tools has had some striking successes and will continue to do so as long as the research stays firmly tied to particular situations.

In the meantime, pure mathematics was not standing still. Very powerful new tools in, for example, the areas of functional analysis, probability theory, asymptotic analysis, numerical analysis and the general theory of differential equations, were being developed, and attempts were begun to apply these ideas to problems of interest to scientists outside of mathematics. There have been some striking successes of this 'modern' style of applied mathematics as well. Unfortunately these two schools of applied mathematics have tended to keep to themselves. Moreover in recent years there has developed a discernible third brand of applied mathematics which seems to incorporate the weak features of both the classical and modern traditions in applied mathematics. It tends to be couched in a lot of generality, but there are no theorems of any mathematical interest, and there are no applications in sight, no particular problem to guide the line of development. Possibly this third style of applied mathematics has developed partly because the classical and modern styles have failed to reach an accord.

In any case, the range of applications has grown enormously, and the growth in methodology, both classical and modern, has not been much slower, thus prompting my initial remark. It seems to me that no good purpose is served by the classical and modern styles of applied mathematics looking down their respective noses at each other. Rather they should join forces, or if that seems too difficult, then at least they shouldn't, by their educational policies, insist that the next generation has to operate with the same prejudices.

Braun's book has aspects that can please both styles of applied mathematicians. The book could perhaps play a role in giving both pure and applied mathematics students and other science students an appreciation of both the classical and the modern styles of applied mathematics, and so far as this is so, the book may make a healthy contribution to the future direction of applied mathematic education.

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Optimization, a theory of necessary conditions, by Lucien W. Neustadt, Princeton Univ. Press, Princeton, New Jersey, 1977, xii + 424 pp., $22.50.

Optimale Steuerung diskreter Systeme, by W. G. Boltjanski, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1976, 326 pp.

The qualitative theory of optimal processes, by R. Gabasov and F. Kirillova, Marcel Dekker, Inc., New York, New York, 1976, xlvi + 640 pp., $55.00.

1. Horreur. "Je me détourne avec éffroi et horreur de cette plaie lamentable des fonctions qui n'ont pas de dérivée"; so said Hermite in a letter to Stieltjes. The reader who shares this aversion to nondifferentiable functions will undoubtedly be affronted by the three books in question. But mathematicians have become much more tolerant about the functions they will talk to. This has been most evident in optimization, where the need to consider differential properties of other than smooth functions arises frequently and fundamentally. In fact, these ill-bred functions are now often brought into the discussion from the start and used systematically, rather than being shunned whenever possible. The extent to which this is true is a striking feature of these three books, all of which were written by well-known researchers in the field of optimal control.

The wedge in this breakthrough was the gradual recognition of the central role in optimization of convexity. This first took place in mathematical programming, and now the methods of convex analysis are being systematically applied in other areas as well; their use in optimal control is currently an active subject for research (see [4]). And convexity *implies* nondifferentiability–not just because differentiability is unnecessary, but because clinging to it is simply not feasible. For example, one of the great successes of convex analysis is duality (see [9]) the pairing with an original minimization problem of a certain closely related maximization problem. Besides being rich in interpretation (e.g. stress vs. reaction, utility vs. price) this concept is at the heart of the most successful computational algorithms in mathematical programming. Yet even if the original problem of interest is smooth, its dual may very well fail to be.

We shall encounter presently some further examples of fundamental nondifferentiability. But before we arrive at what Hermite would think of as this sorry pass, let us look back.