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AFFINE PI RINGS ARE CATENARY

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An affine PI ring, is a ring satisfying a polynomial identity, such that the ring is finitely generated as an algebra over a central subfield k. A long outstanding question in the theory of these rings, was whether any two saturated (increasing) chains of primes beginning at one common prime, must have the same length. (See for e.g. Procesi [3, p. 186].) We answer this question in the affirmative.

THEOREM 1. *If R is a prime affine PI ring then* $\dim R = \dim R/P + \operatorname{ht} P$, *for every prime P.*

Here $\dim R$ means the length of the longest chain of nonzero primes (i.e. Krull dimension), and $\operatorname{ht} P$ is the length of the longest chain of nonzero primes contained in P.

Basic to the proof is passage to the ring $R[T]$, i.e. the ring R with the coefficients of the characteristic polynomials of elements of R adjoined. (R has a central simple quotient ring RK, so think of $\operatorname{tr}(r) \in RK$.) In [4] we proved that $R[T]$ was an integral extension of R, if R is noetherian (in fact a finite R module) and were able to lift our primes to $R[T]$ and prove the result there. If R is not noetherian, we do not have a surjection from Spec $R[T]$ to Spec R, but we are able to lift ht 1 primes.

LEMMA. *If R is prime affine PI and $P \subseteq R$ is a prime with* $\operatorname{ht} P = 1$, *then there exists a P' lying over it in $R[T]$.*

The reason we can do this is $R[T]$ is contained in a finite R module: $k[T]$ is affine and is contained in the complete integral closure of Centre (R) [4, Corollary 2, Theorem 2].

Actually we have a stronger result than in [4], namely:

THEOREM 2. *If R is prime PI and c is the evaluation of a central polynomial, then $c^m R[T] \subset R$ for some m.*

The proof of this is based on the fact that $R[c^{-1}]$ is Azumaya and so $R[T] \subseteq R[c^{-1}]$.

Next we find an irreducible variety W containing $V(P')$, such that W is not

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contained in the complement of $\operatorname{Spec}_n R[T]$, and $V(P')$ is of codimension 1 in W. To do this we proved a generalization of the Going Down Theorem:

THEOREM 3. *If S is a prime ring satisfying a polynomial identity, and is integral over a central subring C, then if $P_1 \subseteq P_2$ are primes in C, and Q_2 is a prime lying over P_2 in S, then there is a prime $Q_1 \subseteq Q_2$ with Q_1 lying over P_1.*

We then show $W = \operatorname{Spec} R$ to complete the proof of the main theorem.

We have also proved in [5]:

THEOREM 4. *If R is a prime PI ring which is a f.g. extension of a right noetherian ring A, then the prime radical of any ideal is nilpotent modulo the ideal.*

This last result was obtained independently by Razmyslov in the case A is a field.

REFERENCES

1. M. Artin, *On Azumaya algebras and finite-dimensional representations of rings*, J. Algebra **11** (1969), 532–563.

2. C. Procesi, *Rings with polynomial identities*, Dekker, New York, 1973.

3. Y. Razymslov, *The Jacobson radical in PI algebras*, Algebra and Logic **13** (1975), 192–204.

4. W. Schelter, *Integral extensions of rings satisfying a polynomial identity*, J. Algebra **40** (1976), 245–257.

5. ———, *Non-commutative affine PI rings are catenary*, Dalhousie University, 1975 (preprint).

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