BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

OPTIMIZATION OF THE NORM OF THE LAGRANGE INTERPOLATION OPERATOR

BY T. A. KILGORE

Communicated by Chandler Davis, December 2, 1976

On an interval $[a, b]$, we may place points $t_0, \ldots, t_n$ such that $a = t_0 < t_1 < \cdots < t_n = b$. Using these points, called nodes, we may construct unique polynomials $y_0, \ldots, y_n$ of degree n, such that, for $1 \leqslant i, j \leqslant n, y_i(t_i) = 1$ and $y_i(t_j) = 0$ for $j \neq i$. The Lagrange interpolating projection on the nodes $t_0, \ldots, t_n$ is the operator which takes any function f continuous on $[a, b]$ to the polynomial $\Sigma_{i=0}^n f(t_i)y_i$. It is easily seen that this projection is bounded for any degree n, for any interval $[a, b]$, and for any set of nodes in $[a, b]$. The norm is easily shown to be the sup norm of $\Lambda = \Sigma_{i=0}^n |y_i|$, called the Lebesgue function of the projection, and thus the norm depends exclusively on the placement of $t_1, \ldots, t_{n-1}$. It is irrelevant, in attempting to minimize the norm, to move t_0 or t_n. Of the function Λ, it is true that $\Lambda(t_i) = 1$ for $0 \leqslant i \leqslant n$, while if $n \geqslant 2$ and if t is not a node, then $\Lambda(t) > 1$. Let $\lambda_1, \ldots, \lambda_n$ be the values given by

$$\lambda_i = \sup_{t \in [t_{i-1}, t_i]} \Lambda(t) \quad \text{for } 1 \leqslant i \leqslant n.$$

Then $\|\Lambda\| = \max_{1 \leqslant i \leqslant n} \lambda_i$.

It was conjectured by Serge Bernstein in 1932 that the norm of the interpolating projection is minimized when the nodes are so placed that $\lambda_1 = \cdots = \lambda_n$, a conjecture rendered plausible, but by no means demonstrated, by the rather obvious fact that

$$\frac{\partial \lambda_i}{\partial t_i} > 0 > \frac{\partial \lambda_{i+1}}{\partial t_i}, \quad \text{for } 1 \leqslant i \leqslant n-1,$$

and by the fact that moving any node into close proximity with one of its neighbors increases $\|\Lambda\|$ without bound. This communication will give the following theorem and an outline of its proof in a series of lemmas.

THEOREM. *For any $n \geqslant 2$, if the norm of the Lagrange interpolation operator on an interval $[a, b]$ with nodes $a = t_0 < t, < \cdots < t_n = b$ is to be minimized, then it is necessary that the local maximum values $\lambda_1, \ldots, \lambda_n$ of the Lebesgue function be equalized.*

The proof of this theorem depends on the fact that $(\lambda_1, \ldots, \lambda_n)$ is a dif-

---

AMS (MOS) subject classifications (1970). Primary 41A10; Secondary 41A50.

ferentiable function of $t_1, \ldots, t_{n-1}$. For $1 \leqslant i \leqslant n$ we denote by X_i the polynomial which agrees with Λ on (t_{i-1}, t_i). We denote by τ_i the (unique) point in (t_{i-1}, t_i) at which $X_i(\tau_i) = \lambda_i$. It is then established that, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n-1$, $\partial\lambda_i/\partial t_j = -y_j(\tau_i)X_i'(t_j)$. Our theorem follows if we can show that, for any position of nodes, every $n-1 \times n-1$ submatrix of $(\partial\lambda_i/\partial t_j)_{ij}$ is nonsingular ($1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n-1$). Using the above expression for $\partial\lambda_i/\partial t_j$, we may with artful cancellations reach the equivalent matrix $(q_i(t_j))_{ij}$, for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n-1$, where $q_i(t) = X_i'(t)/(t-\tau_i)$ is a polynomial of degree $n-2$ or less.[1] Thus each $n-1 \times n-1$ submatrix of $(q_i(t_j))$ is nonsingular if and only if any $n-1$ of $q_1, \ldots, q_n$ form a basis for the space of polynomials of degree $n-2$ or less. In a succession of lemmas, we show that any $n-1$ of $q_1, \ldots, q_n$ are indeed a basis. Here, for reasons of conserving space, the proofs are not given in full.

LEMMA 1. *The polynomials X_1 and X_n have their full complement of n roots on $[a, b]$, and X_1' and X_n' have their full complement of $n-1$ roots on $[a, b]$. For each i, $2 \leqslant i \leqslant n-1$, the polynomial X_i has exactly $n-1$ roots on $[a, b]$, and X_i' has at least $n-2$ roots on $[a, b]$. Each root of each of the above polynomials has multiplicity one, and each root of X_i', $1 \leqslant i \leqslant n$, is a local extremum of X_i.*

PROOF OF LEMMA 1. An obvious counting argument suffices.

LEMMA 2. *For $2 \leqslant i \leqslant n$, X_{i-1}' and X_i' have no common root, nor do X_1' and X_n'.*

PROOF. One uses $X_{i-1} + X_i = 2y_{i-1}$ and an analogous identity for X_1 and X_n. If for example X_{i-1}' and X_i' have a common root, then so does $X_{i-1}' + \alpha y_{i-1}'$, for any real α. One then investigates the number of roots of $X_{i-1} + \alpha y_{i-1}$, when α is chosen so that $(X_{i-1} + \alpha y_{i-1})(r) = 0$, at an assumed common root r of X_{i-1}' and X_i'.

LEMMA 3. *All roots of X_1' and X_n' lie on $[\tau_1, \tau_n]$.*

PROOF. Since X_1' and X_n' have no common root, one can investigate convenient extreme configurations of $t_1, \ldots, t_{n-1}$.

LEMMA 4. *The roots of X_1' and X_n' alternate as we pass from τ_1 to τ_n.*

PROOF. Similar to Lemma 3.

LEMMA 5. *Between τ_1 and the first root of X_{n-1}' appearing on $[\tau_1, \tau_n]$, there is a root of X_n'. The symmetric statement about τ_n, X_2', and X_1' also holds.*

PROOF. Similar to Lemma 3.

LEMMA 6. *On the interval $[\tau_1, \tau_n]$ the roots of $X_1', \ldots, X_n'$ lie in the*

[1] Private communication from Dietrich Braess.

pattern $\hat{1}, n, n-1, \ldots, 3, \hat{2}, 1, n, \ldots, 4, \hat{3}, 2, 1, n, \ldots, 1, n, \widehat{n-1}, n-2, \ldots, 1, \hat{n}$, *when a number i denotes a root of* X_i, $1 \leqslant i \leqslant n$, *and where* $\hat{i}$ *denotes the point* τ_i, $1 \leqslant i \leqslant n$.

PROOF. This proof uses the method of Lemmas 3, 4, 5, and the results of all previous lemmas, and the method of induction.

COROLLARY OF LEMMA 6. *The roots of* $q_1, \ldots, q_n$ *lie in the same locations as those of* $X_1', \ldots, X_n'$, *save that* $\tau_1, \ldots, \tau_n$ *are removed from the list.*

PROOF. Clear.

LEMMA 7. *Without doing harm, we may assume that* $q_i(\tau_1) > 0$ *for* $1 \leqslant i \leqslant n$. *Under this convention, we have* $\operatorname{sgn} q_i(\tau_i) = \operatorname{sgn} q_1(\tau_i)$ *and* $\operatorname{sgn} q_j(\tau_i) = -\operatorname{sgn} q_1(\tau_i)$ *for* $j \neq i$, *where* $2 \leqslant i, j \leqslant n$.

PROOF. Follows easily from Lemma 6 and its Corollary.

LEMMA 8. *The set* $\{q_1, \ldots, q_n\} \sim \{q_k\}$ *is linearly independent for any choice of k,* $1 \leqslant k \leqslant n$.

PROOF. We assume the existence of a nontrivial linear combination $\alpha_1 q_1 + \cdots + \alpha_n q_n = 0$, in which, for some k, $2 \leqslant k \leqslant n$, we have assumed $\alpha_k = 0$, and also $\alpha_1 \geqslant 0$. The sets $N = \{j | \alpha_j < 0 \text{ and } 2 \leqslant j \leqslant n\}$ and $P = \{j | \alpha_j > 0 \text{ and } 2 \leqslant j \leqslant n\}$ are shown nonempty, and the polynomial $N = \alpha_1 q_1 + \Sigma_{j \in N} \alpha_j q_j$ is shown to alternate sign at successive points τ_i, $1 \leqslant i \leqslant n$, thus having at least $n-1$ roots, while having degree no more than $n-2$. The contradiction implies the lemma.

Our theorem now follows from Lemma 8, in light of previous discussion.

ADDED MAY 18, 1977. In the paper which gives details of the above theorem, I complete the proof of Bernstein's conjecture by proving that there is a unique choice of nodes which equalizes the λ_i. I have been informed by C. de Boor and A. Pinkus that they too, using the results of the present note, have completed the proof of Bernstein's conjecture.

DEPARTMENT OF MATHEMATICS, KING ABDULAZIZ UNIVERSITY, MECCA, SAUDI ARABIA