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CLASSIFICATION OF INVOLUTIVE BANACH-LIE ALGEBRAS[1]

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1. The structure and classification theory of semisimple complex Lie algebras is extended to a class of infinite dimensional Banach-Lie algebras. The work abandons the use of a bilinear form, generalizing instead the notion of a compact form.

Following Bonsall and Duncan [1], an operator T on a Banach space is called Hermitian if $|\exp(itT)| = 1$, $t \in \mathbf{R}$. A complex Banach-Lie algebra E with involution * is called symmetric if $\forall x \in E$, $x = x^*$, the operator ad $x \in B(E)$ given by the left regular representation is Hermitian. If E is a symmetric Lie algebra then $\{x \in E: x^* = -x\}$ is a natural analogue of a compact form. A Cartan subalgebra M of E is a maximal selfadjoint abelian subalgebra. Roots are defined as usual: $\alpha \in M'$ is a root of E if the root space $E(\alpha) = \{x \in E: [h, x] = \alpha(h)x \ \forall h \in M\} \neq \{0\}$. The maximality of M implies $E(0) = M$, and for each nonzero root α, $E(\alpha)$ is one dimensional.

A pair (E, M) is called chromatic if E is a semisimple symmetric Lie algebra with $[E, E]$ dense in E, $M \subset E$ is a Cartan subalgebra, and the orbits $G(x)$ in E under the action of the group $G = \{\exp(i\,\mathrm{ad}\,h): h \in M, h = h^*\}$ are relatively compact. Henceforth, (E, M) will always denote an infinite dimensional chromatic pair, and Δ will denote the system of nonzero roots of (E, M).

Harmonic analysis shows that the linear span of all root spaces is dense in E. All results from the finite dimensional root theory carry through for chromatic pairs. A compactness argument on nets of finite dimensional subalgebras shows that two chromatic pairs with isomorphic root systems are algebraically isomorphic. Further, (E, M) has a Chevally form, i.e. there exists $\{x_\alpha, \tau_\alpha: \alpha \in \Delta\}$ such that $x_\alpha \in E(\alpha)$, $x_\alpha^* = x_{-\alpha}$, $\alpha(\tau_\alpha) = 2$ where $\tau_\alpha = [x_\alpha, x_{-\alpha}]$ and $[x_\alpha, x_\beta] = n(\alpha, \beta)x_{\alpha+\beta}$ where $n(\alpha, \beta) \in \mathbf{Z}$. The Cartan integers $\alpha\langle\beta\rangle = \alpha(\tau_\beta)$ are independent of the choice of x_α.

2. Henceforth, (E, M) will be simple (that is, Δ will be indecomposable). Simple chromatic pairs can be classified; they fall into the four big classes A, B, C, D. The proof for the type A or D cases uses ideas due to Kibler (see Kaplansky [4]) but the lack of a bilinear form necessitates modifications. $U \subset \Delta$

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[1] Partial results of author's thesis [3] under J. P. O. Silberstein.

is called a foundation if U is maximal with respect to $\alpha\langle\beta\rangle = 1\ \forall\alpha, \beta \in U, \alpha \neq \beta$, and Δ always contains a foundation of cardinality $\geqslant 9$. Following Kaplansky [4], Δ is an H-system if $\alpha\langle\beta\rangle = 0, \pm 1\ \forall\alpha, \beta \in \Delta, \alpha \neq \beta$. In the sequel, the notations $\mathfrak{gl}(H; C_0)$ etc. follows de la Harpe's [2] usage, where H is a suitable Hilbert space.

THEOREM 1. *Let $U \subset \Delta$ be a foundation of cardinality $\geqslant 6$. If Δ is an H-system, then either*

1. *$\Delta = \Delta_U$, the root system generated by U, in which case (E, M) is of type A, or*
2. *$\Delta = \Delta_{U \cup \{\nu\}}\ \exists\nu \in \Delta\backslash\Delta_U$, in which case (E, M) is of type D.*

The proof depends on Lemma 2, adapted from Kaplansky [4],

LEMMA 2. *If Δ, U are as above and $\nu \in \Delta\backslash\Delta_U$ satisfies $\nu\langle\mu\rangle = 1\ \exists\mu \in U$, then one of the following holds:*

1. *$\nu\langle\eta\rangle = \nu\langle\mu\rangle = 1\ \exists\eta \in U, \eta \neq \mu; \nu\langle\xi\rangle = 0\ \forall\xi \in U, \xi \neq \eta, \mu$,*
2. *$\nu\langle\eta\rangle = 0\ \exists\eta \in U, \eta \neq \mu; \nu\langle\xi\rangle = 1\ \forall\xi \in U, \xi \neq \eta$;*

and proceeds through a case-by-case study of possible root strings to show that Δ is isomorphic to the root system of $\mathfrak{gl}(H; C_0)$ or $\mathfrak{o}(H, J_{\mathrm{R}}; C_0)$, and hence (E, M) is of type A or D respectively.

THEOREM 3. *Let $U \subset \Delta$ be a foundation of cardinality $\geqslant 9$, and let $\Gamma \subset \Delta$ be a maximal sub H-system such that $U \subset \Gamma$. If Δ is not an H-system, then either*

1. *$\exists\alpha \in \Delta\backslash\Gamma$ such that $\alpha\langle\eta\rangle = 2\ \exists\eta \in U$, in which case (E, M) is of type C, or*
2. *$\exists\alpha \in \Delta\backslash\Gamma$ such that $\eta\langle\alpha\rangle = 2\ \exists\eta \in U$, in which case (E, M) is of type B.*

The proof depends on

LEMMA 4. *Let Δ, Γ, U be as above.*

1. *If $\alpha \in \Delta\backslash\Gamma$ satisfies $\alpha\langle\eta\rangle = 2\ \exists\eta \in U$, then either*
   (a) *$\alpha\langle\mu\rangle = 2\ \forall\mu \in U$, or*
   (b) *$\alpha\langle\mu\rangle = 0\ \forall\mu \in U, \mu \neq \eta$, in which case $\alpha'\langle\mu\rangle = 2\ \forall\mu \in U$, where $\alpha' = 2\eta - \alpha$.*
2. *If $\alpha \in \Delta\backslash\Gamma$ satisfies $\eta\langle\alpha\rangle = 2\ \exists\eta \in U$, then either*
   (a) *$\mu\langle\alpha\rangle = 2\ \forall\mu \in U$, or*
   (b) *$\mu\langle\alpha\rangle = 0\ \forall\mu \in U, \mu \neq \eta$, in which case $\mu\langle\alpha'\rangle = 2\ \forall\mu \in U$, where $\alpha' = \eta - \alpha$;*

and shows via a case-by-case analysis that Δ is isomorphic to the root system of $\mathfrak{sp}(H, J_{\mathrm{Q}}; C_0)$ or $\mathfrak{o}(H, J_{\mathrm{R}}; C_0)$ and hence that (E, M) is of type C or B respectively.

Thus any simple chromatic pair is the completion, in a suitable norm, of a classical simple involutive Lie algebra of operators on a Hilbert space.

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