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THE EXISTENCE AND UNIQUENESS OF A SIMPLE GROUP GENERATED BY {3, 4}-TRANSPOSITIONS

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Recently Fischer [1] discovered three finite simple groups each of which contains a conjugacy class D of involutions such that for all x and y in D the order of the product xy is 1, 2, or 3. Such a class is called a class of 3-transpositions. More generally, if π is a set of positive integers and D is a conjugacy class of involutions in the finite group G, then D is said to be a class of π-transpositions in G if D generates G and for all noncommuting elements x and y of D the order of xy is in π. Fischer has produced evidence suggesting the existence of a new simple group containing a class of {3, 4}-transpositions. Fischer determined a number of properties of the group, including its order, which is $2^{41}3^{13}5^6 7^2 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$ or approximately 4.15×10^{33}. However, the questions of the existence of such a group and the uniqueness of its isomorphism type remained unanswered.

We have now constructed a simple group G having the properties specified by Fischer and in addition we have shown that G is determined, up to isomorphism, by certain of these properties. A description of the 13,571,955,000 {3, 4}-transpositions in G has been obtained and the action of a set of generators for G on these transpositions has been determined. The details of the construction and the proof of uniqueness, which involve extensive use of a computer, will appear elsewhere.

If H is any group, then $Z(H)$ will denote the center of H, H' the commutator subgroup of H, and $O_2(H)$ the largest normal 2-subgroup of H. If h is an element of H and K is a subgroup of H, then $C_K(h)$ is the centralizer in K of h and h^K is the set of K-conjugates of h.

Let L be a perfect 2-fold covering group of the simple group ${}^2E_6(2)$. That is, $L' = L$, $|Z(L)| = 2$, and $L/Z(L)$ is isomorphic to ${}^2E_6(2)$. These conditions determine L up to isomorphism. In Aut(L) there is a unique conjugacy class of involutions σ centralizing a subgroup of L isomorphic to $Z_2 \times F_4(2)$. Let E denote the split extension of L by $\langle \sigma \rangle$ and let d generate $Z(E)$.

The smallest of the Fischer simple groups generated by 3-transpositions has order $2^{17}3^9 5^2 7 \cdot 11 \cdot 13$ and is usually denoted F_{22}. It is known that

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$|\mathrm{Aut}(F_{22}): F_{22}| = 2$ and that $\mathrm{Aut}(F_{22})$ is generated by a unique class of $\{3, 4\}$-transpositions. In E there are exactly two conjugacy classes of subgroups isomorphic to $\mathrm{Aut}(F_{22})$, which are interchanged by an automorphism of E acting trivially on L. Fix a subgroup S of E isomorphic to $\mathrm{Aut}(F_{22})$. In S' choose two noncommuting 3-transpositions d_2 and d_2' and choose a $\{3, 4\}$-transposition d_3 in S. Set $K = C_S(d_2) \cap C_S(d_2')$. Then $|K : K'| = 4$ and K' is isomorphic to PSU(4, 3). There is a unique subgroup Q of index 2 in $O_2(C_E(d_2))$ which is normalized by K. Define five subgroups of E as follows

$$E_1 = E,$$
$$E_2 = C_E(d_2),$$
$$E_3 = C_E(d_3) \simeq Z_2 \times Z_2 \times F_4(2),$$
$$E_4 = S,$$
$$E_5 = KQ.$$

The group $E_2/\langle d \rangle$ is a split extension of an extra-special group of order 2^{21} by a group of the form $\mathrm{PSU}_6(2) \cdot Z_2$.

We can now state our main result.

THEOREM. *There exists a simple group G, unique up to isomorphism, such that*

(1) $|G| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47.$

(2) *G contains E and $C_G(d) = E$.*

(3) *Under conjugation by E, the class $D = d^G$ breaks up into five classes D_i, $1 \leqslant i \leqslant 5$.*

(4) *For $1 \leqslant i \leqslant 5$ there is an element x_i in D_i such that $C_E(x_i) = E_i$.*

The class D is a class of $\{3, 4\}$-transpositions in G. In fact $D_1 = \{d\}$, $D_1 \cup D_2 \cup D_3$ is the set of elements in D commuting with d, and elements of D_4 and D_5 have products of order 3 and 4 with d respectively. It can be shown that $x_2 = d_2$ and $x_3 = d_3$.

REFERENCE

1. B. Fischer, *Finite groups generated by 3-transpositions*. I, Invent. Math. 13 (1971), 232–246. MR 45 #3557.

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