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THE EXISTENCE OF MINIMAL IMMERSIONS OF TWO-SPHERES

BY J. SACKS AND K. UHLENBECK[1]

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In this article we announce a series of results on the existence of harmonic maps from surfaces to Riemannian manifolds and, as corollaries of these results, obtain theorems on the existence of minimal immersions of 2-spheres.

Let N be a compact connected Riemannian manifold and, for convenience, assume that N is isometrically imbedded in $\mathbf{R}^k$ for some sufficiently large k. Let M be a closed Riemann surface with any metric compatible with its conformal structure. A map $s \in L_1^2(M, \mathbf{R}^k) \cap C^0(M, N)$ is called harmonic if it is an extremal map of the energy integral

$$E(s) = \int_M |ds|^2 \, d\mu_M = \int_M \operatorname{trace} I(x) \, d\mu_M$$

where

$$I(x) = \sum_{i=1}^{k} ds^i \otimes ds^i(x) \in T_x^*(M) \otimes T_x^*(M).$$

Harmonic maps satisfy an Euler-Lagrange equation

$$\Delta s + A(s)(ds, ds) = 0$$

in a weak sense, where A is the second fundamental form of the imbedding $N \subset \mathbf{R}^k$. It then follows from regularity theorems that harmonic maps are C^∞. If s is harmonic and a conformal immersion, it is also an extremal for the area integral.

Proving the existence of harmonic maps of M into N by direct methods from global analysis such as Morse theory or Ljusternik-Schnirelman theory applied to E defined on some function space manifold is difficult, because E is invariant under the conformal group of M, and the extremal maps of E form a noncompact set when $M = S^2$. In particular, E does not satisfy condition C of Palais-Smale. However, for $\alpha > 1$, a slightly different integral,

$$E_\alpha(s) = \int_M (1 + |ds|^2)^\alpha \, d\mu_M$$

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for $s \in L_1^{2\alpha}(M, \mathbf{R}^k) \cap C^0(M, N) = L_1^{2\alpha}(M, N)$, is C^2 and satisfies the Palais-Smale condition C in a complete Finsler metric on $L_1^{2\alpha}(M, N)$. If we normalize the area of M to be 1 then, as $\alpha \longrightarrow 1$, $E_\alpha(s) \longrightarrow E(s) + 1$. By examining the convergence of a sequence s_α of critical maps of E_α as $\alpha \longrightarrow 1$, various results on the existence of harmonic maps are obtained.

MAIN CONVERGENCE THEOREM. *Let $s_{\alpha(i)}$ be a sequence of critical maps of $E_{\alpha(i)}$, $\alpha(i) \geqslant 1$, $\lim_{i\to\infty} \alpha(i) = 1$. Then there exist a subsequence i', a harmonic map $s: M \longrightarrow N$ and a finite number of points $\{x_1, \ldots, x_l\}$ such that $s_{\alpha(i')} \longrightarrow s$ in $C^1(M - \{x_1, \ldots, x_l\}, N)$. Moreover, there exist l nontrivial harmonic maps $\tilde{s}_k: S^2 = \mathbf{R}^2 \cup \{\infty\} \longrightarrow N$, $k = 1, 2, \ldots, l$, such that for $x \in \mathbf{R}^2$, $\tilde{s}_k(x) = \lim_{i'\to\infty} s_{\alpha(i')}(x_k + \rho_{i',k}x)$ where $\lim_{i'\to\infty} \rho_{i',k} = 0$. Note that $l = 0$ is possible.*

In the proof of the convergence theorem, we make use of the following extension theorem.

EXTENSION THEOREM. *Let D denote the open unit disk. Let $s: D - \{0\} \longrightarrow N$ be a harmonic map defined on $D - \{0\}$. If $E(s) < \infty$, then s extends to a smooth harmonic map $\tilde{s}: D \longrightarrow N$.*

The convergence theorem is used to obtain a series of results on harmonic maps. The first result applies to the case $M \neq S^2$. Every free homotopy class in $C^0(M, N)$ induces a map from $\pi_1(M)$ into $\pi_1(N)$, which is defined only up to conjugation by an element of $\pi_1(N)$, due to the lack of base point.

THEOREM 1. *There exists a minimizing harmonic map among all maps inducing the same conjugacy class of maps from $\pi_1(M)$ to $\pi_1(N)$.*

COROLLARY. *If $\pi_2(N) = 0$, then there exists a minimizing harmonic map in every homotopy class of maps in $C^1(M, N)$.*

In the case $\pi_2(N) \neq 0$, $\pi_1(N)$ acts on $\pi_2(N)$ by moving the base point around representatives of elements. Given an element Γ in the free homotopy classes in $C^0(S^2, N)$, we associate with Γ a subset $\pi_1(\Gamma) \subset \pi_2(N)$ consisting of all based homotopy classes formed by connecting the spheres in Γ with the base point in N. Thus $\pi_1(\Gamma)$ is an orbit of $\pi_1(N)$ in $\pi_2(N)$.

THEOREM 2. *There exists a set of free homotopy classes $\Lambda_i \subset C^0(S^2, N)$ such that elements $\lambda_i \in \pi_1(\Lambda_i)$ generate the group ring $\pi_2(N)$, and each Λ_i contains a minimizing harmonic map $s_i: S^2 \longrightarrow N$.*

In general there may be no nontrivial minimizing harmonic maps. However, we do have the following special result.

THEOREM 3. *If the universal covering space of N is not contractible, then there exists at least one nontrivial harmonic map $s: S^2 \longrightarrow N$.*

There is a close relationship between harmonic maps and minimal surfaces. If U is an open set in M and an immersion $s: U \longrightarrow N$ is conformal, then s is harmonic if and only if $s(U)$ is minimal. Given any harmonic map $s: M \longrightarrow N$, we define

$$w(z) = |s_y(z)|^2 - |s_x(z)|^2 + 2i(s_x(z), s_y(z))$$

where $z = x + iy$ is a local isothermal coordinate chart on M. Let $\phi(z) = w(z)dz^2$. From the Euler-Lagrange equations for the energy integral, one can show that if s is harmonic then $\phi(z)$ is a holomorphic quadratic differential. Therefore, $\phi(z) = 0$ if $s: S^2 \longrightarrow N$ is harmonic. A more general theorem applies to all surfaces M.

THEOREM 4. *If s is a critical map of E, where the variation is taken over both the map s and the conformal structure on M, then the holomorphic quadratic differential ϕ associated with s is identically zero.*

Since there is only one conformal structure on S^2, we obtain the result that if $s: S^2 \longrightarrow N$ is harmonic, then s is a minimal immersion except at points z_0 with $s_x(z_0) = s_y(z_0) = 0$.

THEOREM 5. *If $s: S^2 \longrightarrow N$ is harmonic, then s is a conformal branched immersion and $s(S^2)$ is a minimal surface except at the branch points of s.*

COROLLARY. *The maps $s_i: S^2 \longrightarrow N$ in the statement of Theorem 2 can be taken to be minimal branched immersions.*

MAIN THEOREM. *Let N be a C^∞ compact Riemannian manifold of dimension $\geqslant 3$ such that the universal covering space of N is not contractible. Then there exists a nonconstant C^∞ map $s: S^2 \longrightarrow N$ such that $s: S^2 - \{x_1, \ldots, x_l\} \longrightarrow N$ is a conformal minimal immersion and $x_1, \ldots, x_l$ are branch points of s.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, CHICAGO CIRCLE, CHICAGO, ILLINOIS, 60680