$$(12) \qquad \sum_{\substack{n \leqslant x \\ n \in \mathcal{P}_k}} a_n g(n), \qquad x \to \infty, k = 1, 2, \ldots,$$

under some general assumptions on $(a_n)_{n=1}^{\infty}$. In (12), g denotes a suitable weighting function, and $\mathcal{P}_k$ the set of square-free integers having exactly k prime factors. Although one of Bombieri's assumptions is usually not easy to verify for given $(a_n)_{n=1}^{\infty}$, there is no doubt that his work is an important contribution to our knowledge of general sieve methods, which is likely to influence their future development.

Hooley begins the chapters of his book with a historical survey on the relevant problem, and ends them with a discussion of other applications of the method or of possible relaxations of the hypothesis used. This practice is helpful to the reader and provides a good orientation of the subject. The book is written with great attention to detail. It affords an insight into the richness of the problems which can successfully be treated with the help of sieve methods. It can be recommended to anybody interested in sieve methods.

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Abstract analytic number theory, by John Knopfmacher, North-Holland Mathematical Library, vol. 12, North-Holland, Amsterdam & Oxford; American Elsevier, New York, 1975, ix + 322 pp., $29.50.

The reader may wonder what the title of Knopfmacher's book signifies. The word "abstract" refers to an axiomatic set-up of the material which is treated here within the framework of *arithmetical semigroups*, the standard example being the positive integers with their multiplicative structure. The word "analytic" refers to the admission of analytic functions and Cauchy's theorem as tools in proving theorems. Finally "number theory" indicates that this work arose from generalizations of theorems on ordinary integers.

The main topics treated in this book are rooted in:

(i) Dirichlet's theorem that there are infinitely many primes in every residue class l modulo k, if the positive integers k and l are relatively prime;

(ii) the prime number theorem that

$$\pi(x) \sim x/\log x, \qquad x \to \infty,$$

where $\pi(x)$ denotes the number of primes less than x;

(iii) the result of G. H. Hardy and S. Ramanujan that

$$p(n) \sim \exp\{n^{1/2}(\pi\sqrt{2/3} + o(1))\}, \qquad n \to \infty,$$

where $p(n)$ denotes the number of unrestricted partitions of the positive integer n;

(iv) S. Ramanujan's series expansion

$$\sigma(n) = \frac{\pi^2 n}{6} \sum_{k=1}^{\infty} \frac{c_k(n)}{k^2}, \qquad n = 1, 2, \ldots,$$

where $\sigma(n)$ denotes the sum of all positive divisors of n and

$$c_k(n) = \sum_{\substack{0<l<k \\ (l,k)=1}} e^{2\pi i n l/k}.$$

Results which sharpen (i)–(iii) considerably are now known. Also, statements analogous to (i)–(iv) hold in a more general context. We shall describe briefly the developments to which (i)–(iv) have given rise.

Dirichlet proved (i) in 1837. The prime number theorem (ii) was first proved by J. Hadamard and C. de la Vallée Poussin, independently, in 1896. In 1899 C. de la Vallée Poussin obtained a substantial improvement of (i) and (ii). He proved that there is a positive c such that for positive, relatively prime integers k and l,

$$(1) \qquad \pi(x; k, l) = \frac{1}{\varphi(k)} \int_2^x \frac{dy}{\log y} + O(x \exp(-c\sqrt{\log x})), \qquad x \to \infty,$$

where φ denotes the Euler function and $\pi(x; k, l)$ the number of primes less than x lying in the residue class l modulo k. Since we have

$$\int_2^x \frac{dy}{\log y} \sim \frac{x}{\log x}, \qquad x \to \infty,$$

(1) obviously implies

$$(2) \qquad \pi(x; k, l) \sim x/\varphi(k)\log x, \qquad x \to \infty,$$

and, in particular, (ii), if we choose $k = l = 1$. It also follows from (1) that the primes are evenly distributed among the $\varphi(k)$ residue classes which are relatively prime to k.

In 1900, D. Hilbert gave mathematical investigations a great stimulus, when he delivered his famous list of problems at the International Congress of

Mathematicians in Paris. In his eighth problem Hilbert first recalls the Riemann hypothesis that all the nonreal zeros of the Riemann zeta function have real part $\frac{1}{2}$ and Goldbach's conjecture that every even integer greater than 3 is a sum of two primes. In the second part of his eighth problem he suggests the problem of carrying over the results on the distribution of ordinary prime numbers to results on the distribution of prime ideals in an algebraic number field.

Although the Riemann hypothesis and Goldbach's conjecture are still unsettled, many results connected with Hilbert's eighth problem have already been proved.

Information on the zero-free region of the Riemann zeta function has been obtained by the use of delicate estimates for exponential sums. The strongest results so far proved rely on a method introduced by I. M. Vinogradov in 1935. These results have led, on their part, to improved error terms in (1) which, however, are still far from $O(x^{1/2}\log x)$ that is equivalent to the truth of the generalized Riemann hypothesis for Dirichlet's L-series. It has also turned out that for many arithmetical questions a better knowledge of the error term of (1) in its dependence on k and l would be useful. The most important results known in this direction are the Siegel-Walfisz theorem (1936) and the theorem of E. Bombieri and A. I. Vinogradov (1965). The former asserts that (1) holds uniformly with respect to k and l in the range $0 < l < k$, $(k,l) = 1$, $1 \leqslant k \leqslant \log^{\alpha} x$, where α is any positive number. The latter says that for any $A > 0$ there is a $B > 0$ such that for $K = x^{1/2}\log^{-B} x$,

$$\sum_{1\leqslant k\leqslant K} \max_{2\leqslant y\leqslant x} \max_{\substack{l \\ (l,k)=1}} \left| \pi(y;k,l) - \frac{1}{\varphi(k)}\int_2^y \frac{du}{\log u} \right| = O(x\log^{-A} x), \qquad x\to\infty. \tag{3}$$

While these two theorems can be used, e.g., to prove results somewhat weaker than Goldbach's conjecture, the latter sometimes even serves as a good substitute for the generalized Riemann hypothesis for all Dirichlet's L-series. To get an idea of its power, one should observe that even if the error term in (1) is $O(x^{1/2}\log x)$, this does not lead to an essentially better estimate for the sum in (3).

As far as the second part of Hilbert's eighth problem is concerned, the first contribution is due to E. Landau, who proved the analogue of (ii) for algebraic number fields in 1903. After E. Hecke's work on zeta functions (1917-1920), analogues of (1) were accessible by standard procedures in any algebraic number field. Moreover, the theorems obtained in this transition process did not only give new information on the distribution of prime ideals, but also contained new results on ordinary prime numbers. Such results were obtained, e.g., by Hecke (1920) using his zeta functions with Grössencharacters and by N. Čebotarev (1926) using Artin's L-functions. Hecke proved as a special case that for $-1/\sqrt{2} \leqslant \alpha < \beta \leqslant 1/\sqrt{2}$ there is a positive number $C_{\alpha\beta}$ such that

$$\# \{p \text{ prime} | p = u^2 - 2v^2 \leqslant x; u, v \text{ integral}; \alpha \leqslant v/u \leqslant \beta\} \sim C_{\alpha\beta} x/\log x, \qquad x \to \infty,$$

where $\# M$ denotes the cardinality of the set M. Čebotarev showed that for any conjugacy class C of the Galois group G of a finite Galois extension L over the rationals $\mathbf{Q}$,

$$\# \{p \text{ prime} | p \leqslant x; p \text{ unramified for } L/\mathbf{Q} \text{ and } F_{L/\mathbf{Q}}(p) = C\} \sim (\# C/\# G) \cdot x/\log x, \qquad x \to \infty,$$

where $F_{L/\mathbf{Q}}(p)$ denotes the Frobenius conjugacy class determined by p. This theorem is an extension of (2), to which it reduces in the case of the cyclotomic field $L = \mathbf{Q}(e^{2\pi i/k})$.

A generalization of (ii) different from that proposed by Hilbert originated with A. Beurling in 1937. While Hilbert suggested an algebro-arithmetical extension of (ii), Beurling was interested in the analytic nature of the prime number theorem. He started with a sequence $P = (p_j)_{j=1}^{\infty}$ of so-called *generalized primes*, i.e. with real numbers p_j satisfying $1 < p_1 \leqslant p_2 \leqslant \cdots$ and having no finite accumulation point. Denoting by π_P and N_P the counting functions of the generalized primes and generalized numbers, namely

$$\pi_P(x) = \# \{p_j | p_j \leqslant x\},$$

$$N_P(x) = \# \{(\alpha_1, \alpha_2, \ldots) | p_1^{\alpha_1} p_2^{\alpha_2} \cdots \leqslant x, \ \alpha_j \text{ nonnegative integral for } j = 1, 2, \ldots\},$$

he looked, in particular, for functions $\varepsilon(x)$ tending to 0 for $x \to \infty$ such that the assumption

$$N_P(x) = Ax(1 + \varepsilon(x)), \qquad x \to \infty, \tag{4}$$

for some positive number A implies

$$\pi_P(x) \sim x/\log x. \tag{5}$$

If $\varepsilon(x) = O(\log^{-\gamma} x)$ for $x \to \infty$, he found that (5) holds for γ greater than $\frac{3}{2}$, and that (5) can no longer be inferred from (4) in the case of $\gamma = \frac{3}{2}$. His results contain not only (ii) but also Landau's prime ideal theorem proved in 1903. For if P is constituted by the norms of the prime ideals in an algebraic number field, (4) holds with an ε even satisfying

$$\varepsilon(x) = O(x^{-\theta}), \qquad x \to \infty, \tag{6}$$

for some positive θ. Beurling obtained his results by refining earlier proofs of the prime number theorem in a subtle way.

The notion of generalized primes being so extensive, it may happen, on the other hand, that the asymptotic behaviour of π_P is much easier to discuss than that of N_P. This happens, e.g., in the case when

$$P = (y^j)_{j=1}^{\infty}, \qquad y > 1,$$

where we have

$$\pi_P(y^x) = \sum_{1 \leqslant j \leqslant x} 1 \sim x, \qquad x \to \infty, \tag{7}$$

and

$$N_P(y^x) = \sum_{1 \leqslant j \leqslant x} p(j).$$

In view of (iii) it is obvious that

$$N_P(y^x) \sim \exp\{x^{1/2}(\pi\sqrt{2/3} + o(1))\}, \qquad x \to \infty. \tag{8}$$

Hardy and Ramanujan proved (iii) using a tauberian argument in 1917. A year later they improved it with the help of the circle method to

$$p(n) = \left(e^{C\lambda_n}/4\sqrt{3}\,\lambda_n\right)(1 + O(1/\lambda_n)), \qquad n \to \infty,$$

where $C = \pi\sqrt{2/3}$ and $\lambda_n = (n - 1/24)^{1/2}$. Finally in 1937, H. Rademacher gave an exact expression for $p(n)$, namely

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} k^{1/2} A_k(n) \frac{d}{dn}\left(\frac{\sinh(C\lambda_n/k)}{\lambda_n}\right), \tag{9}$$

where $A_k(n)$ denotes a finite sum of roots of unity. These results sharpen (8) considerably. A proof of Rademacher's formula is still rather involved. It is intimately connected with the theory of automorphic forms, whereas for (8) it is sufficient to know (7). A result similar to (8) can therefore be proved for any sequence P of generalized primes satisfying something like (7).

An identity for an arithmetical function, which is much easier to obtain than (9), is given in (iv). Together with some other identities of a similar kind, it was proved by S. Ramanujan in 1918. Later it became apparent that such expressions are related to almost periodic functions. An extensive theory of *Ramanujan expansions*, i.e. expansions like (iv) in terms of the sums c_k, has been developed in recent years. It is much in the spirit of the general theory of almost periodic functions. Ramanujan expansions of multiplicative functions have attracted a special interest. If the identity

$$c_k(n) = \sum_{\substack{d|n \\ d|k}} d\mu(k/d),$$

where μ denotes the Moebius function, is taken as the definition of c_k, a theory of Ramanujan expansion can also be developed for generalized numbers satisfying a condition similar to (4).

Since Beurling's work, many authors have investigated, under suitable conditions, various properties of generalized primes. Such an axiomatic point of view is also taken in the work under review. The author uses the concept of arithmetical semigroups which is equivalent to the notion of Beurling's generalized primes. Knopfmacher starts with a long discussion of arithmetical semigroups and the functions defined thereon. According to Knopfmacher, he hopes to convince the reader of the usefulness of the axiomatic set up by giving many "concrete" examples fitting in the "abstract" approach he has chosen. These examples range from arithmetic and algebra to geometry and topology. For instance arithmetical semigroups are associated to the category of finite abelian groups or to the category of compact, simply connected, globally symmetric Riemannian manifolds. This can be done mainly because a "unique factorization theorem" holds in the corresponding categories. Knopfmacher then proves (5) under conditions (4) and (6) with the help of the tauberian theorem of Wiener and Ikehara. Similarly, an analogue of (2) is given in an axiomatic set up, however only by assuming a statement which already plays a crucial role in the proof of Dirichlet's theorem (i). For one of his axioms amounts in the classical case to the assumption that Dirichlet's L-series formed with real nontrivial characters do not vanish at $s = 1$. Using the method of Hardy and Ramanujan developed in 1917, Knopfmacher derives an asymptotic result of type (8) for generalized primes satisfying something like (7). Under some further conditions he also proves analogues of (iii). Another chapter of his book is devoted to the theory of Ramanujan expansions on arithmetical semigroups. Finally he ends with a rather extensive bibliography. As an addendum, A. G. Postnikov's book, *Introduction to the analytic theory of numbers* [Moscow, 1971 (Russian)], may be suggested. Among many other things, analogues of (i)–(iii) are treated within the setting of generalized primes in Postnikov's book.

Since Knopfmacher's book assumes only a moderate mathematical background and the proofs are modelled along classical lines, the book may serve as an introduction to analytic number theory itself. However, the reader may find the author's practice of using the same notation for the coefficients of a Dirichlet series as well as the Dirichlet series itself rather unusual. This practice gives rise to statements such as "... the constant function ... is called the zeta function" (p. 36).

It is not the author's purpose to prove the best results known about the questions he treats. He has endeavoured rather to illustrate his theorems with a large number of examples. In particular, he has paid attention to the explicit calculation of the numerical constants in the asymptotic results for many special cases.

The author indicates his view of the future development in the theory of arithmetical semigroups with a list of open problems. Moreover, since "concrete" analytic number theory is a living part of mathematics, further possibilities of proving its theorems in an axiomatic setting cannot be excluded. When such developments are carried out, however, it may be worthwhile to keep in mind the following, somewhat free, quotation from H. Weyl (*Gesammelte Abhandlungen*, Vol. I, p. 393): *Methods of this general nature should achieve that which no special approach is capable of doing, namely reveal the common features of a large complex of phenomena.* The reviewer has missed an adequate stressing of this view of general methods in Knopfmacher's book.

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