notation for the range of an extension of a Gödel numbering is carried along in a way which can really be a nuisance to a reader. The prose unfortunately does not share the clean elegance of the mathematical development. Finally, it is surprising that Springer-Verlag did not catch some carelessness with proper names, e.g. "Weierstrauss".

But these minor matters aside, Monk has brought together an enormous range of interesting material. He has written an important and valuable book which will be a standard reference for some time to come.

A few errata: p. 19, line 8, the subscript 0 should be 1; p. 31, line 13, "on" should be "an"; p. 81, line -9, the word "recursive" is (crucially) missing after "binary"; p. 267, in the proof of (*), the list of finite structures should repeat each one infinitely often; p. 350, line 9, the subscript on *Fmla* should be $\mathfrak{L}$; p. 442, line 10, *Fmla* should have the superscript $\mathfrak{n}$ (in addition to its subscript).

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

Connections, curvature, and cohomology. Vol. I: *de Rham cohomology of manifolds and vector bundles*, by Werner Greub, Stephen Halperin and Ray Vanstone, Academic Press, New York and London, 1972, xix + 443 pp., \$27.50.

Connections, curvature, and cohomology. Vol. II: *Lie groups, principal bundles, and characteristic classes*, by Werner Greub, Stephen Halperin and Ray Vanstone, Academic Press, New York and London, 1973, xxi + 541 pp., \$35.00.

Connections, curvature, and cohomology. Vol. III: *Cohomology of principal bundles and homogeneous spaces*, by Werner Greub, Stephen Halperin and Ray Vanstone, Academic Press, New York, San Francisco, London, 1976, xxi + 593 pp., \$49.50.[1]

The topic of these volumes, relations between the topology and the differential geometry of manifolds, in particular, the notion of "characteristic classes", has occupied mathematicians for a long time. The first instances are probably Gauss's expression for the linking number of two curves by a double integral; and Dyck's theorem $\int_S K\,dA = 2\pi\chi_S$, where S is a closed surface, K the Gauss curvature and χ_S the Euler characteristic (1888, for a surface in 3-space; later proved (by Blaschke?) intrinsically, with Gauss's Theorema Egregium and the Gauss-Bonnet formula). The latter theorem is still *the* model for the present topic.

Another important example is Hopf's theorem $\sum j_P = \chi_M$, where the j_P are the indices of the zeroes of a vector field V on the closed manifold M, and χ_M again the Euler characteristic; there is also its earlier companion: The

[1] These books are included as volumes in the Pure and Applied Mathematics Series of Academic Press. Vol. I is numbered 47-I, Vol. II is 47-II, and Vol. III is 47-III.

curvatura integra of a closed hypersurface M in $\mathbf{R}^n$, n odd, is $\frac{1}{2}\chi_M$. The Stiefel-Whitney classes (characteristic cohomology classes attached to systems of vector fields in a manifold) belong here. (However, they do not appear in the text, since the discussion is restricted to real coefficients.)

All this is, so to speak, the classical period; a new period began in 1940, when Allendoerfer and Fenchel, independently, found the long-sought generalization of Dyck's theorem: *There is a universal polynomial in the coefficients of the Riemann curvature tensor of a Riemannian manifold M^n, n even, that happens to be a function on M and whose integral gives χ_M.*

Soon after this Pontryagin and Chern discovered the characteristic classes that now carry their names, cohomology classes attached to the differentiable or complex structure of a manifold. In both cases one of the possible definitions involves expressions in differential forms related to the curvature tensor; it is this aspect that came to be expanded greatly. In 1949 it was realized what was going on (A. Weil): The proper objects to look at are principal G-bundles P, with G a Lie group; thus G acts freely on P, and the quotient P/G is a manifold M. (The prime example: M a Riemannian manifold, P the collection of orthonormal frames in the tangent spaces at the various points of M; $G = O(n) =$ orthogonal group; an orthogonal matrix (a_{ij}) sends a frame $\{v_i\}_1^n$ to the new frame $\{w_j = \sum a_{ij} v_i\}_1^n$.) The theory of connections (parallel displacement) of differential geometry, as extended to this situation by Ehresmann and others, attaches to a connection ω in P (they exist) a curvature, expressed as a $\mathfrak{g}$-valued ($\mathfrak{g} =$ Lie algebra of G) differential 2-form Ω on P (and equivariant: $\Omega(vg, wg) = \operatorname{Ad} g^{-1} \cdot \Omega(v, w)$, with Ad the adjoint action of G on $\mathfrak{g}$). The fundamental idea now is to take any symmetric multilinear (degree r) function φ on $\mathfrak{g}$ that, moreover, is invariant under Ad (in brief: an invariant polynomial) and to substitute the curvature form Ω for the variables of φ, interpreting multiplication as exterior. It turns out, somewhat miraculously, that the $2r$-form $\varphi(\Omega, \ldots, \Omega)$, so obtained, is a form on M (and not only on P) and that it is closed, thus defining via de Rham a cohomology class of M; and, moreover, this class is independent of the choice of connection. (To prove all this takes time.) The homomorphism from the algebra of invariant polynomials on $\mathfrak{g}$ to the cohomology $H^*(M)$ so defined is known as the Weil map; the image is the algebra of *characteristic classes* of P. Thus the invariant polynomials on $\mathfrak{g}$ form the "universal source" of all characteristic classes of all principal G-bundles. (For $G = O(n)$, $\mathfrak{g} =$ skew $n \times n$ matrices, the invariant (under $X \to MXM^{-1}$) polynomials form a polynomial algebra generated by the "characteristic coefficients", the coefficients of $t^{n-2}, t^{n-4}, \ldots$ in the characteristic polynomial of the generic matrix X in $\mathfrak{g}$ [the coefficients of $t^{n-1}, t^{n-3}, \ldots$ are 0]. These generators become the Pontryagin classes under the Weil map. For $G = SO(2m)$ the last coefficient, the determinant, is the square of another invariant, the Pfaffian; the corresponding cohomology class, under the Weil map, is called the Euler class. For the frame bundle of M one gets this way the Allendoerfer-Fenchel integrand.)

(From the topological point of view the characteristic classes are the image of the cohomology of a classifying space, base space of a universal (contractible) bundle, from which any bundle can be induced. In the first instance characteristic classes serve to distinguish different principal G-bundles over M. They become more important in the theory of cobordism [R. Thom], where manifolds are considered modulo the relation of being a boundary, the fundamental tool being the evaluation of [top dimensional] characteristic classes on the fundamental cycle of the manifold. In this context other, exotic, cohomology theories appear, going beyond the frame work of the present volumes.)

In the meantime the theory of Lie groups had taken a new turn with Hopf's work: The notion of primitive cohomology class had appeared (these are the elements a in $H^*(G)$ with $\mu^*(a) = a \otimes 1 + 1 \otimes a$, where μ: $G \times G \to G$ is the product in G). And one has the theorem that $H^*(G)$, for compact connected G, (which according to E. Cartan can be identified with the Ad-invariants in the exterior algebra $\wedge \mathfrak{g}^*$; $\mathfrak{g}^*$ = dual of $\mathfrak{g}$) can be written as $\wedge P_{\mathfrak{g}}$, the exterior algebra over the space $P_{\mathfrak{g}}$ of primitive elements. Combining all this with the Weil map led to an algebraic generalization of connections (Weil, H. Cartan, Koszul, and others). The leading idea is to replace any principal G-bundle P, or, more generally, any manifold on which G acts, by its algebra of differential forms (a similar step was taken recently by D. Sullivan in his theory of rational homotopy type) and thus to consider arbitrary differential algebras R (graded-commutative, with $a^2 = 0$ if degree a odd); the action of the (connected) group G on P gets translated into operators $\theta(X)$ (Lie derivative) and $i(X)$ (substitution) on R, for X in $\mathfrak{g}$, with suitable relations (e.g., the homotopy formula $\theta(X) = d \circ i_X + i_X \circ d$). The base manifold appears as the subalgebra $R_{\theta,i}$ of R formed by the elements that are invariant under (= nullified by) all $\theta(X)$ and $i(X)$. A connection is a linear map from $\mathfrak{g}^*$ to R^1 (degree 1 part of R), suitably related to θ and i. If a connection exists, then curvature Ω can be defined. The Weil map becomes a homomorphism from $(\vee \mathfrak{g}^*)_\theta$, the θ-invariants in the symmetric algebra $\vee \mathfrak{g}^*$, to $H^*(R_{\theta,i})$. An extensive development, involving a lot of homological algebra, spectral sequences and Koszul complexes (complexes attached to modules over polynomial rings whose virtue is that they replace whole spectral sequences by consideration of a single differential), and difficult to summarize, sets in now. The *Weil algebra* $W(\mathfrak{g})$ is the tensor product of $\vee \mathfrak{g}^*$ and $\wedge \mathfrak{g}^*$, with suitable i, θ, d. There are two main facts: (1) the cohomology of $W(\mathfrak{g})$ (and also that of the subalgebra $W(\mathfrak{g})_\theta$ of the θ-invariants) is 0; and (2) the Weil map extends to a "classifying" map of $W(\mathfrak{g})$ into any R as above, preserving operators. They make $W(\mathfrak{g})$ the analog of the universal bundle in bundle theory. There is the notion of suspension or Cartan map from $(\vee \mathfrak{g}^*)_\theta$ to $(\wedge \mathfrak{g}^*)_\theta$: x, in the former, is dy in $W(g)_\theta$; project y to $(\wedge \mathfrak{g}^*)_\theta$. (This way of going from base to fiber had appeared, e.g., in Chern's elegant intrinsic proof of the Allendoerfer-Fenchel result.) A "reverse" map, transgression, goes from the primitives, $P_{\mathfrak{g}}$, to $(\vee \mathfrak{g}^*)_\theta$; in the process one proves the important theorem that for a reductive $\mathfrak{g}$ the invariants $(\vee \mathfrak{g}^*)_\theta$ *always* form a polynomial algebra (example: $O(n)$ or $SO(2m)$ above). (Hilbert's "First Main Theorem" only gave finite generation.) The next big result says that transgression in $W(\mathfrak{g})_\theta$ and the Weil map determine the cohomology of the θ-invariants R_θ (translated to geometry this means $H^*(P)$, G assumed compact). All this gets applied to the study of quotient spaces G/H, i.e., of the relative cohomology $H^*(\mathfrak{g}, \mathfrak{h})$ of the Lie algebras. The first main result identifies this cohomology with that of the Koszul complex $(\vee \mathfrak{h}^*)_\theta \otimes \wedge P_{\mathfrak{g}}$, relative to a differential determined by transgression and the restriction $(\vee \mathfrak{g}^*)_\theta$ to $(\vee \mathfrak{h}^*)_\theta$. Starting from these general facts, one can describe $H^*(\mathfrak{g}, \mathfrak{h})$ very concretely in terms of its image in the cohomology of $\mathfrak{g}$ (under the projection $G \to G/H$) and the image of the Weil map $(\vee \mathfrak{h}^*)_\theta \to H^*(\mathfrak{g}, \mathfrak{h})$, by setting up universal diagrams that relate the various entities. There is a number of interesting special cases: G and H of equal rank, symmetric spaces, ... To spell out an example a bit: there is the case "H noncohomologous to 0 in G", i.e., the restriction map $H^*(\mathfrak{g}) \to H^*(\mathfrak{h})$ is surjective. This is proved equivalent (under suitable reductivity assumptions) to either: the projection k^*: $H^*(\mathfrak{g}, \mathfrak{h}) \to H^*(\mathfrak{g})$ is injective; or: the Weil map $(\vee \mathfrak{h}^*)_\theta \to H^*(\mathfrak{g}, \mathfrak{h})$ is trivial ("no characteristic classes"); or: $H^*(\mathfrak{g}, \mathfrak{h})$ is generated, as algebra, by odd-dimensional elements; or several other properties. Similar facts are developed for the G/H-bundles over a manifold M, derived from principal G-bundles P by forming P/H. (A topological approach to the whole theory appears in Borel's work on classifying spaces.)

All of this material (much of it quite difficult to find in the literature) and a lot more has been organized in the three volumes under review into a clear, coherent and complete account. Everything needed is developed from scratch (necessarily sometimes in rather condensed form); very rarely is a fact brought in from the outside, only "linear algebra", some real analysis, and basic topology are assumed. No single step is ever difficult (however, there are many steps and one has to keep many definitions and symbols in mind); one usually just defines a linear map (often a derivation), or extends one (possibly multiplicatively, symmetric or exterior), or restricts one, or induces one. Definitions sometimes have to be taken on faith; motivation may come a good deal later. There are a lot of clever arguments and improvements of earlier approaches (e.g., Poincaré duality is proved with almost no work). Cohomology is *defined* in terms of differential forms on manifolds (homology appears only incidentally, although fully explained at that point). A minor warning: The symplectic group $Sp(n)$ is denoted by either $Sy(n)$ or $Q(n)$.

The first volume consists of an excellent account of the basic facts on manifolds: vector bundles, differential forms, Poincaré duality (for noncompact manifolds as nonsingularity of the pairing between cohomology and compact cohomology), de Rham's theorem (via the cohomology of the nerve of a simple cover), degree of a map, Hopf's theorem on maps $M^n \to n$-sphere, a thorough discussion of integration over the fiber in a bundle, sphere bundles and the Euler class, the Thom isomorphism, Hopf's theorem on vector fields and Lefschetz's coincidence theorem, ...

The second volume, after an exposition of Lie group theory, introduces principal and associated bundles, connections (principal and linear), parallel displacement, covariant derivative, and curvature; and then discusses the concrete-geometric case of the Weil map, from the point of view of principal bundles and also from that of vector bundles with given structure tensors. One finds there the cohomology of the classical groups (the exceptional groups do not appear) and of some homogeneous spaces, the formulae for the characteristic classes (Pontryagin, Euler, Chern), and Chern's proof for the Gauss-Bonnet-Dyck-Allendoerfer-Fenchel-Weil-Chern theorem.

The third volume, after introductory material on spectral sequences and (very welcome) on Koszul complexes, gives a thorough and complete treatment of the algebraic form of the Weil map. Many examples, classical groups and homogeneous spaces, are worked out. A minor quibble: The third volume does not have an index of notations.

There is a large number of interesting problems in the first two volumes, ranging from simple illustrations to rather difficult general theorems, and adding a lot of "general mathematical education". There is a very extensive bibliography. The third volume has a set of interesting notes on the history of the various facts, and on relations with other topics (e.g., the currently active area of characteristic classes of foliations).

The authors have done us a real service in making this fascinating, but rather complex, field accessible and organizing it so clearly and competently.

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

Applications of sieve methods to the theory of numbers, by C. Hooley, Cambridge Tracts in Mathematics, no. 70, Cambridge University Press, Cambridge, London, New York, Melbourne, 1976, xiv + 122 pp., $18.95.

In number theory there are famous conjectures which can easily be explained even to a layman, but which still resist a complete solution. Two of them are as follows.

There exists an infinity of primes p such that $p + 2$ is also a prime (*the twin prime problem*).

Every even integer greater than 3 is a sum of two primes, or equivalently, every integer greater than 5 is a sum of three primes (*Goldbach's problem*).

It is in the attempt to solve such problems that sieve methods have been developed. The first steps were taken by V. Brun around 1920. Since his pioneering work, there has been progress in refining the techniques and improving the results of sieve theory. The power of the elementary methods originally used has been considerably increased by the combination of