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Rational methods in Lie algebras, by George B. Seligman, Lecture Notes in Pure and Applied Math., vol. 17, Marcel Dekker, New York and Basel, 1976, viii + 346 pp., $29.50.

The simple Lie algebras over **C** were classified by W. Killing and E. Cartan just prior to 1900. These comprise four infinite families A_n, B_n, C_n, D_n, along with five exceptional algebras E_6, E_7, E_8, F_4, G_2, the subscript denoting the rank. The infinite families occur as Lie algebras of the "classical" complex Lie groups, e.g., A_n is the Lie algebra of the special linear group $\mathrm{SL}(n+1, \mathbf{C})$. The exceptional Lie algebras are intimately related to certain other exceptional structures arising in nonassociative algebra, such as 8-dimensional Cayley algebras (octonions) and nonspecial 27-dimensional simple Jordan algebras.

In a nutshell, the classification amounts to characterizing each simple Lie algebra L by a finite configuration of vectors in a euclidean space $\mathbf{R}^n$ (the *root system*, consisting of certain linear functions on a Cartan subalgebra of L), subject to some constraints on the angles between vectors; the determination of all possible root systems is then a sophisticated exercise in euclidean geometry. Of course, one also has to prove the existence of algebras of all types A–G.

Actually, it is the *real* Lie groups and their Lie algebras which lie closer to the sources of Lie theory in differential geometry. If L is a simple Lie algebra over **R**, there are just two possibilities:

(1) The complexification $L_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} L$ may be a direct sum of two simple Lie algebras, interchanged by complex conjugation, in which case L is isomorphic to either of these viewed as Lie algebra over **R**; the classification of these real simple algebras leads back directly to the list A–G.

(2) $L_{\mathbf{C}}$ may be simple, in which case L is called a *real form* of $L_{\mathbf{C}}$. Here the type A–G of $L_{\mathbf{C}}$ is called the *split type* of L; each split type turns out to embrace two or more distinct possibilities for L.

By 1914 Cartan [**4**] had obtained the complete classification of simple Lie algebras over **R**, from a "compact" point of view, exploiting the fact that each simple Lie algebra over **C** has a unique compact real form (corresponding to a compact Lie group). Later there were other treatments in this vein by F. Gantmacher [**5**], S. Murakami [**10**], as well as treatments from an "anticompact" point of view, cf. S. Araki [**2**], I. Satake [**11**] (appendix by M. Sugiura). In either case, the essential idea is to compare the unknown L with the known $L_{\mathbf{C}}$. This amounts to a determination of the nonabelian galois cohomology set $H^1(\mathrm{Gal}(\mathbf{C}/\mathbf{R}), \mathrm{Aut}\, L_{\mathbf{C}})$.

It was not long before attempts were made to classify simple Lie algebras over other fields of characteristic 0, motivated by the fact that the classical linear groups (special linear, orthogonal, symplectic) exist over general fields. In the 1930's, W. Landherr [**8**], [**9**], a student of E. Artin, made some progress in this direction. N. Jacobson, H. Freudenthal, and others pushed the purely algebraic study of Lie algebras substantially farther and enlarged the study of the exceptional types [**7**]. (The paper [**6**] furnished the title of the book under review, though it was concerned only with structural questions, not classification.)

As in the real case, one can exploit the fact that a given simple Lie algebra L over a field F "splits" over a finite galois extension K of F, i.e., L_K is a direct sum of one or more split simple Lie algebras of types A–G constructed just as in the complex case. (The point is that the simple Lie algebras over $\mathbf{C}$ are constructible over $\mathbf{Q}$, indeed over $\mathbf{Z}$, and therefore have natural analogues over arbitrary fields.) Again the crucial problem is to determine the F-forms of a simple Lie algebra over K; this classification may be viewed as a problem in galois cohomology, though the splitting field is perhaps not so natural as in the case of $\mathbf{C}/\mathbf{R}$.

The most comprehensive classification scheme occurs in the work of J. Tits **[14]** (cf. **[12]** for further details), for simple algebraic groups over arbitrary fields. This too is done in the spirit of galois descent, and incorporates the various earlier contributions to the subject. This set-up includes as a special case the classification of (noncompact) simple Lie algebras over fields of characteristic 0, but is not limited to such fields. The price paid for this generality is much greater technical sophistication than one needs for the study of the Lie algebras alone, where linear algebra is the main tool.

Seligman's monograph provides a relatively self-contained account of the classification of (noncompact) simple Lie algebras over a field F of characteristic 0. The story is complete except for some cases in rank 1 still under study by B. N. Allison, J. R. Faulkner, and others. The feature which sets the author's treatment apart from all those mentioned above is his emphasis on working exclusively over F–hence the "rational" in the title–without recourse to a splitting field. Instead of classifying the F-forms of the split algebras of types A–G, he relies entirely on the "relative" root system and builds up the algebras systematically from rational data. (He does, however, use some known properties of split algebras to study certain naturally occurring split subalgebras of the algebras in question.) His approach has the advantage of leading rather directly to the explicit construction of Lie algebras. Most of the ingredients are of course implicit in the existing literature, though scattered; the author especially emphasizes his indebtedness to the work of Tits.

In order to convey the spirit of the author's approach, it is necessary to get a little more detailed. Chapter I begins with a simple (or slightly more general) Lie algebra L over F. Choose a maximal split toral subalgebra T of L: "toral" means that ad t is a semisimple endomorphism for each $t \in T$, while "split" means that all eigenvalues of all ad t lie in F. It follows readily that T is abelian, allowing L to be written as the direct sum of common eigenspaces. $L = L_0 + \Sigma L_\alpha$, where α runs over a finite set Σ of nonzero linear functions on T ("roots"). In case L is already split, T is a Cartan subalgebra and this is the usual root space decomposition, with $L_0 = T$ and $\dim L_\alpha = 1$. But in general, T could be 0 (and Σ empty). Then L is called *anisotropic*, which is the same as "compact" if $F = \mathbf{R}$. Take for example the Lie algebra of the special orthogonal group of an anisotropic quadratic form over F.

Because the nature of anisotropic Lie algebras depends heavily on F, uniform results can be obtained only if L is assumed to be *isotropic* ($T \neq 0$), which will be done from now on. The author shows that Σ is a *root system* in the usual (Bourbaki) sense, though not necessarily reduced: α, 2α can both be roots (this happens already when $F = \mathbf{R}$). Because L is simple, Σ is irreducible and therefore has type A–G (if reduced) or BC_n (if nonreduced).

The study of the internal structure of L now proceeds much as in the split case, relying heavily on the Killing form and on the representation theory of split 3-dimensional simple subalgebras associated with suitable triples $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$, $[e_\alpha e_{-\alpha}] \in T$. The author goes on to show that all maximal split toral subalgebras are conjugate by inner automorphisms of L. All of this is closely parallel to the Borel-Tits theory of simple (or, more generally, reductive) algebraic groups over an arbitrary field [3], and can in fact be viewed as a special case of that theory.

The decomposition $L = L_0 + \Sigma L_\alpha$ may still be rather unwieldy, since the root spaces L_α can be enormous. Chapter III examines the structure of L from another angle, suggested by a construction of Tits [13]. Choose a set of simple roots in Σ (or, in the nonreduced case, a set of simple roots in the reduced root system Σ' consisting of those $\alpha \in \Sigma$ for which 2α is not a root). Construct a 3-dimensional split simple subalgebra for each simple root α as indicated above, and call S the subalgebra of L generated by all of these. S is split, simple, of the same type A–G as L (or of type C_n if L is of type BC_n), with T as a Cartan subalgebra. In particular, a lot is known about S. (For a construction of this sort in algebraic groups, cf. [3, §7].)

Now the idea is to decompose L as an ad S-module. Irreducible S-modules are characterized by their highest weights relative to T. Since the weights here belong to $\Sigma \cup \{0\}$, there are at most four types of irreducible summands, corresponding to the weight 0 (trivial S-module), the highest root (adjoint S-module S), the highest short root, if two root lengths occur (call the module M'), and half the highest root, if the type is BC_n (call the module M'').

To simplify, consider just the case of single root length (types A, D, E). Here $L = D \oplus (S \otimes A)$, where D is a trivial S-module and where A is a vector space over F, whose dimension counts the number of copies of the adjoint S-module present. But A is not just a vector space. Close scrutiny of the way in which D and $S \otimes A$ interact under the Lie bracket leads to the introduction of an algebra structure on A, with D acting on A by derivations. For example, when the type of L is A_1, A turns out to be a Jordan division algebra over F with $D \cong \mathrm{Der}(A)$. The algebra corresponding to $E_{7,1}^{78}$ in Tits' list [14] arises this way: S is 3-dimensional, A is a 27-dimensional exceptional Jordan division algebra, D is of dimension 52 (anisotropic simple Lie algebra of split type F_4), and L itself is of split type E_7, agreeing with the dimension calculation $133 = \dim L = 52 + 3 \cdot 27$. (Even though the author is ignoring the split types of his algebras, it would have been helpful to label them in accordance with Tits' list, to facilitate comparisons.)

It is somewhat remarkable that L can be built up in this way from scratch. What makes the "coordinatization" work is the very limited number of choices available at each step. For example, bilinear pairings of S to F are just multiples of the Killing form, because $\dim \mathrm{Hom}_S(S \otimes S, F) = 1$. The author works out in an appendix all the relevant dimensions (involving F, S, M', M''), using the known facts about multiplicities of irreducible S-modules in tensor products; this allows him to exhibit a basis for each Hom space. Of course, this degree of explicitness requires a series of lengthy case-by-case calculations.

The algebra discussed above is exceptional in a strong sense, since it does not exist over $\mathbf{R}$, or over p-adic or number fields. Another example (with $F = \mathbf{R}$) may be illuminating. When L has type A_2, there are several natural candidates for L: the derived algebras of full 3×3 matrix algebras over $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ (the quaternions). But there is a further choice for L: $L = D \oplus (S \otimes \mathbf{O})$, where S is the split algebra of type A_2 and dimension 8, $\mathbf{O}$ is the Cayley division algebra (octonions) of dimension 8, and $D = \mathrm{Der}(\mathbf{O})$ is a compact Lie algebra of split type G_2, having dimension 14. Thus $\dim L = 78$, consistent with the fact that L is a real form of the split algebra E_6 (listed by Tits as ${}^1E_{6,2}^{28}$).

Chapter V transforms the somewhat abstract coordinatization of Chapter III into more familiar realizations of the various simple Lie algebras, following a brief discussion of centroids and central simple algebras. This chapter ends by recovering Cartan's list of real forms. The inner automorphism groups of the various algebras, generated by all exp ad X (X nilpotent), are also realized concretely. Earlier, in Chapter II, the author proves the simplicity of these groups, following the method of Tits (BN-pairs).

Chapter IV, which uses some of the information in Chapter III, provides rational criteria for isomorphism of simple Lie algebras over F, by a refinement of the method of B. N. Allison [**1**], which still required some use of field extensions. Roughly speaking, the "anisotropic kernels" (coming from the centralizers of maximal split toral subalgebras) and the respective root spaces must correspond in a way compatible with Weyl group action. The precise statement of the isomorphism theorem is too complicated to reproduce here.

Only about a third of the book consists of "theory", concentrated in the brief Chapters I, II, IV, and the beginning of III. The rest of the space is given over to coordinatization and realization of the individual types. This material, by its nature, requires extensive calculation (from which the author does not shrink) and leads to an uncomfortable density of symbols. A certain amount of dedication is required before one tackles (e.g.) p. 127. On the other hand, the theory makes for interesting reading, and it is easy enough to locate any specific type of algebra about which one wants to know more.

The present volume is an expanded and revised, but still not entirely polished, version of a mimeographed set of Yale lecture notes issued in 1969. (Chapter IV is a notable recent addition.) The author apologizes in his Foreword for the minor inconsistencies in notation and references which remain, and compensates by providing a helpful "reference guide" to the main results. Probably the publisher should apologize for the rather steep price, considering the soft cover and the photographic reproduction from typescript, which does not enhance readability. In any event, Seligman's monograph will not appeal to the casual reader, but definitely will be a valuable resource for those who (like the reviewer) find simple Lie algebras to be a fascinating phenomenon.

REFERENCES

1. B. N. Allison, *Isomorphism of simple Lie algebras*, Trans. Amer. Math. Soc. **117** (1973), 173–190. MR **48** #6194.

2. S. Araki, *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ. **13** (1962), 1–34. MR **27** #3743.

3. A. Borel and J. Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150. MR **34** #7527.

4. E. Cartan, *Les groupes réels simples, finis et continus*, Ann. Sci. École Norm. Sup. (3) **31** (1914), 263–355.

5. F. Gantmacher, *On the classification of real simple Lie groups*, Mat. Sb. **5** (1939), 217–249. MR **2**, 5.

6. N. Jacobson, *Rational methods in the theory of Lie algebras*, Ann. of Math. (2) **36** (1935), 875–881.

7. ______, *Exceptional Lie algebras*, Dekker, New York, 1971. MR 44 #1707.

8. W. Landherr, *Über einfache Liesche Ringe*, Abh. Math. Sem. Univ. Hamburg **11** (1935), 41–64.

9. ______, *Liesche Ringe vom Typus A über einem algebraischen Zahlkörper (Die lineare Gruppe) und hermitesche Formen über einem Schiefkörper*, Abh. Math. Sem. Univ. Hamburg **12** (1938), 200–241.

10. S. Murakami, *Sur la classification des algèbres de Lie réelles et simples*, Osaka J. Math. **2** (1965), 291–307. MR **34** #1460.

11. I. Satake, *Classification theory of semi-simple algebraic groups*, Dekker, New York, 1971. MR **47** #5135.

12. M. Selbach, *Klassifikationstheorie halbeinfacher algebraischer Gruppen*, Bonner Math. Schriften **83** (1976).

13, J. Tits, *Une class d'algèbres de Lie en relation avec les algèbres de Jordan*, Indag. Math. **24** (1962), 530–535. MR **26** #3753.

14. ———, *Classification of algebraic semisimple groups*, Proc. Sympos. Pure Math. vol. 9, Amer. Math. Soc., Providence, R. I., 1966, pp. 33–62. MR 37 #309.

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Differential topology, by Morris W. Hirsch, Springer-Verlag, New York, x + 221 pp.

The study of manifolds is the central theme of topology, and provides the overall motivation. In the early period, which ended around 1940, the methods used were rather intuitive and geometric. One school viewed the manifold primarily as a combinatorial object; algebraic topology developed out of this approach. The other tried to work directly with the differential structure of the manifold; for example, de Rham's theory of differential forms and Morse's theory of calculus of variations in the large. The importance of Morse's approach does not seem to have been fully appreciated at the time, and from the mid-thirties to the mid-fifties was a period of relative neglect for the differential viewpoint. Algebraic topology was progressing by leaps and bounds, during this period, but was little concerned with manifolds as such. It was not until the mid-fifties that it was seen how to use the powerful new techniques which had been discovered to obtain results of a kind which would have excited Poincaré and other pioneers. One such result was undoubtedly Milnor's discovery of the exotic spheres. Another was Thom's theory of cobordism, the first serviceable classification of manifolds to be obtained. These outstanding successes led to a great resurgence of interest in the study of manifolds and the modern phase of differential topology got under way. Hirsch's book is not so much concerned with the