H. S. Bear that the Gleason parts of the spectrum of a function algebra are determined by mapping into an appropriate convex set K and showing that the Gleason parts are just the inverse images of the sets of the Nikodym decomposition of K.

Part IV looks at the effects of choosing an ordered field other than R. Part V returns to linear spaces over R to compare these algebraic-geometric operations in L with more usual topologies for L. The book ends with some account of the natural topology in L.

The authors have surveyed and digested the literature of this topic quite thoroughly. This set of lecture notes gives interested mathematicians a very full account of the kind of topological structure forced on a linear space by its scalar field.

M. M. DAY

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Model theoretic algebra: *Selected topics*, by Greg Cherlin, Lecture Notes in Mathematics, vol. 521, Springer-Verlag, Berlin and New York, 1976, iv + 232 pp., $9.50.

In an address to the International Congress of Mathematicians at Cambridge, Massachusetts in 1950, Abraham Robinson pointed out that "contemporary symbolic logic can produce useful tools–though by no means omnipotent ones–for the development of actual mathematics, more particularly for the development of algebra and, it would appear, of algebraic geometry." A similar observation was made by Alfred Tarski in an address to the same Congress in which he defined some of the basic notions of that branch of logic which is now called model theory–that is the study of the properties of mathematical structures expressible in formal mathematical languages.

That the expectations of these two giants of model theory were more than fully realized in the succeeding decades is indicated by the scope of the volume under review, which is an exposition of selected results in the model theory of such diverse algebraic systems as groups, rings, modules, fields, division rings, ordered fields and valued fields. Not all of the results presented are applications of model theory to algebra in the strict sense that they are theorems expressed in conventional algebraic terms and proved by model-theoretic methods; but many of the others are applications in the broader sense that they show how–in the words of Robinson in a later paper **[9]**–"certain basic facts and notions of Algebra, for example the notion of an algebraically closed field, can be placed and generalized within the framework of Model Theory."

The book under review, which consists of lecture notes of a course given by the author at M.I.T. in 1974 and again at the University of Heidelberg in 1975, constitutes an expeditious and extensive introduction to the burgeoning field of "model theoretic algebra." The author is a knowledgeable and informative guide, who provides a broad view of the subject, never losing sight of the forest for the trees. (The reverse side of the same coin, however, is that the reader is often given only a very sketchy map to find his way through the trees: a good deal of effort is required on the part of the reader to fill in all the details of proofs.) The book is successfully organized around "a few main themes which were championed by the late Professor Abraham Robinson. In particular we lay great stress on the role played by transfer theorems and existentially complete structures in algebra." The model theory dealt with in the book is almost exclusively "classical" model theory, which deals with statements of the first-order, or lower, predicate calculus; these are, roughly, finitary expressions built up from relation and function symbols using variables, constants, logical connectives ("and", "or", "not", "implies") and quantifiers ("for all", "there exists"); the statements are "first-order" in that variables are understood to stand for elements of the domain of the structure and not, for example, for subsets of the domain. (The author provides a brief summary of the basic notions of model theory in Chapter 0, but some previous familiarity with these ideas is an almost essential prerequisite for reading this book.)

A typical example of a transfer theorem is the following, formulated in Robinson's 1950 paper. Let φ be a first-order statement about fields; then φ is true in some algebraically closed field of characteristic zero if and only if φ is true in every algebraically closed field of characteristic zero if and only if there is an integer n such that φ is true in all algebraically closed fields of finite characteristic $p > n$. The first equivalence is a precise but weak form of the heuristic Lefschetz's Principle of algebraic geometry; it is weak in that most theorems of algebraic geometry are not naturally formulated as statements of the lower predicate calculus[1]. Nevertheless, this transfer principle has interesting consequences; one elementary but notable one escaped attention until 1967 when it was observed by James Ax: an injective morphism of an algebraic variety into itself is surjective.

A striking early success of the model theoretic approach to algebra was Abraham Robinson's solution of Hilbert's Seventeenth Problem, solved originally by Artin and Schreier. Robinson's method (discussed in Chapter I of Cherlin's book) provides a beautifully transparent solution based on an important concept from model theory, that of model completeness. A class Σ of algebraic structures is called *model complete* if the members of Σ satisfy the following transfer principle: whenever $\mathfrak{A}$ and $\mathfrak{B}$ belong to Σ and $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, then any first-order statement about $\mathfrak{A}$–which may refer to elements of $\mathfrak{A}$–is true in $\mathfrak{A}$ if and only if it is true in $\mathfrak{B}$. The fact that the class of real closed fields is model complete may be proved in a number of ways, all depending ultimately on the crucial algebraic fact that the real closure of

[1] Other precise versions of Lefschetz's Principle, which come closer to capturing the content of the informal original, have been given by logicians using formal languages stronger than the lower predicate calculus. The reviewer's version is discussed in Cherlin's article **[2]**.

an ordered field is unique up to isomorphism.[2] Once model completeness is proved, the solution to Hilbert's problem follows quickly by a kind of mathematical pun. A similar sort of argument can be employed to prove the Hilbert Nullstellensatz, using the fact that the class of algebraically closed fields is model complete[3].

The greatest achievement of model-theoretic algebra thus far is probably the Ax-Kochen-Ershov result on Artin's conjecture for $\mathbf{Q}_p$, the field of p-adic numbers. As Cherlin emphasizes in Chapter II, the main theorem may be viewed as a transfer principle for Hensel fields, from which one obtains in a special case the fact that the ultraproducts $\pi_p \mathbf{Q}_p/D$ and $\pi_p F_p((t))/D$ satisfy the same first-order statements. (Here F_p is the field of order p and D is a nonprincipal ultrafilter over the set of primes.) The known Diophantine properties of $F_p((t))$–which inspired Artin's conjecture–then lead to the truth of Artin's conjecture "almost everywhere" for $\mathbf{Q}_p$, a best possible result in view of Terjanian's counterexample. Cherlin sketches the proof of the transfer theorem but refers for detail to the excellent exposition by Kochen [**5**]. Cherlin does discuss Kochen's p-adic analogue of Artin-Schreier Theory, which is based on the model completeness of the class of "p-adically closed" fields.

The class of real closed fields intuitively bears the same relation to the class of ordered fields as the class of algebraically closed fields does to the class of fields. This analogy can be made precise by means of model theory; indeed the subclass in each case is the class of structures which are existentially complete with respect to the larger class. A structure $\mathfrak{A}$ in a class Σ is called *existentially complete* with respect to Σ if whenever $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B} \in \Sigma$, then any "existential statement"–i.e. a statement which in prenex form involves only existential quantifiers–about $\mathfrak{A}$ is true in $\mathfrak{A}$ if and only if it is true in $\mathfrak{B}$. That the formalization of this analogy is more than just an idle exercise is indicated by the fact that it led Robinson to the definition of a differentially closed field, which has turned out to be a valuable concept in differential algebra and one whose study continues to make important use of model-theoretic methods. (A useful introduction to this subject, which is not discussed by Cherlin, is given in the article [**12**].)

While the class of algebraically closed fields is axiomatizable by first-order statements and the same is true for real closed fields and differentially closed fields, the situation is quite different in the case of existentially complete

[2] Artin-Schreier's original proof of this fact was based on Sturm's Theorem. Cherlin gives as an exercise to prove the uniqueness theorem using only the Intermediate Value Theorem–a problem which defeated the efforts of a number of notable mathematicians. He provides a terse hint, but for a complete proof one may consult the paper of H. Gross and P. Hafner in Comment. Math. Helv. **44** (1969), 491–494. A proof using only the Intermediate Value Theorem was also given by James Ax in 1966 but was apparently never published by him.

[3] Robinson also obtained results about bounds in Hilbert's Nullstellensatz and Hilbert's Seventeenth Problem. For information on this subject, which is not discussed in the book, see Robinson's article [**9**] or the reviewer's survey article [**3**]. A valuable expository article on the general subject of model completeness is A. Macintyre's [**6**].

(noncommutative) groups or division rings. (Existentially complete groups are usually called "algebraically closed".) Just how different is made clear in the investigations carried out by A. Macintyre and W. Wheeler using a combination of model-theoretic and recursion-theoretic methods (and relying heavily on the algebraic results of Paul M. Cohn in the case of division rings). The key model-theoretic tool, the forcing construction, due to A. Robinson–inspired by Paul J. Cohen's set theoretic notion–is discussed in Chapter III of Cherlin and the application to division rings and groups in Chapter IV. (See also [4].) As an example of the power of the forcing method, we mention one of its early successes: the proof by Macintyre (solving a problem posed by the asomatous Eli Bers) that there are countable algebraically closed groups which do not satisfy the same first-order statements; the forcing technique is crucial here in that it provides a method of construction concrete enough to yield distinguishable algebraically closed groups.

Chapter V treats the theory of existentially complete modules, due to G. Sabbagh, E. Fisher and the reviewer. Here, model theory sheds light on the notions of injective and pure-injective and gives a central position to an algebraic condition, weaker than noetherian, called coherence.

The last two chapters of the book provide brief introductions to other areas of active investigation in model-theoretic algebra. Chapter VI deals with the first-order model theory of abelian groups, which is well understood thanks to the work of W. Szmielew. (Cherlin's presentation follows a later approach, due to E. Fisher and the reviewer, which is based on an analysis of the structure of saturated groups.) But the use of infinitary languages and of set-theoretic ideas continues to be a fruitful method of studying infinite abelian groups. The final chapter, on $\aleph_1$-categorical fields, affords a glimpse of the significant role which stability theory, a major area of research in pure model theory (see [**11**]), has come to play in the study of algebraic systems from a model-theoretic point of view.

(Two errata: on p. 131, a stronger version of Q' is needed to make D' existentially complete; on p. 225, reference 45 is due to Schreier not Schilling. There are a number of other typographical errors which will not cause difficulty.)

To sum up, the book under review is a well-written, wide-ranging survey of a field which connects mathematical logic and algebra with benefits to both. The list of references which follows consists of expository articles and books which supplement the material contained in this book.

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PAUL C. EKLOF

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Pattern synthesis, lectures in pattern recognition, Volume 1, by U. Grenander, Applied Mathematical Sciences, No. 18, Springer-Verlag, New York and Berlin, 1976, vii + 509 pp., $14.80.

A generation or so ago, applied mathematics consisted primarily of the solution of partial differential equations subject to diverse geometrical constraints. The development of computers radically altered this emphasis: numerical methods superseded analytical approximations, and the rapidly expanding capacities of digital computers obviated the need for ingenious shortcuts.

Analog computers, with their versatile internal structures, would probably have given rise to investigations into structural configurations if they had not fallen behind digital automata because of their restricted speeds. Nevertheless, the design of digital computer circuitry itself has stimulated work on configurational mathematics, and the very scalar quality of the digital computer has necessitated an analytical approach to pattern recognition. The theory of self-organizing automata (Turing), the genetic code and research on neural nets have proven the power of an algorismatic approach to structure.

A pattern is an ordered array whose components bear a well-defined relation to each other. Pattern recognition amounts to the identification of these components, and the nature of their interrelationships. Such identification is to a certain extent subjective: the expression of the internal vibrations of material objects as a series of monochromatic mutually orthogonal functions is convenient, but not necessarily fundamental in acoustic theory. The description of a crystal in terms of stacked cubical unit cells may please the crystallographer, but the solid-state chemist will prefer a model based on stacking spheres of different diameters. Thus the same crystal will appear in different "gestalten" in different contexts.

A very effective way of reducing the apparent complexity of a pattern is the above-mentioned algorismatic approach, namely by synthesizing it according