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Fuzzy sets and their applications to cognitive and decision processes, edited by L. A. Zadeh, K.-S. Fu, K. Tanaka and M. Shimura, Academic Press, New York, 1975, x + 496 pp., $16.00.

Introduction to the theory of fuzzy subsets. Volume I, by A. Kaufmann (Translated from the French edition of 1973), Academic Press, New York, 1975, xvi + 416 pp., $22.50.

Applications of fuzzy sets to systems analysis, by C. V. Negoiţă and D. A. Ralescu (Revised and translated from the Romanian edition of 1974), Birkhäuser Verlag and Halsted Press (Wiley and Sons, Inc.), Basel and New York, 1975, 191 pp., $19.75.

Scientists have long sought ways to use the precision of mathematics to tame the imprecisions of the real world. One may see many-valued logic, topology, and probability theory as different attempts to be precise about imprecision. In 1965, Lotfi Zadeh [1] suggested that the proper tool for handling imprecision was to replace the rigid all-or-none of set membership by *graded* membership–so that the characteristic function $\chi_A: X \to \{0, 1\}$ of a set in the universe X was to be replaced by a membership function $\chi_A: X \to [0, 1]$ with weights falling in the interval $[0, 1]$. Set operations then generalize as follows:

$$\chi_{A\cup B}(x) = \max[\chi_A(x), \chi_B(x)], \qquad \chi_{A\cap B}(x) = \min[\chi_A(x), \chi_B(x)],$$

$$\chi_{\bar{A}}(x) = 1 - \chi_A(x).$$

While the pointwise operations were already contained in the literature of multivalued logics [2], Zadeh's, call for their wider application initiated a growth industry of which the three books under review are but partial expression. The burden of my review is the following: I admire Professor Zadeh's efforts to explore the manifold uses of graded membership and find a number of the papers in this literature intriguing but I am depressed by the number of authors who seem to have accepted sets with graded membership as *the* only way to describe the world's fuzziness in mathematical terms. Let me develop this theme by first examining a number of articles in the first of the volumes under review, which contains 19 papers from a U.S.-Japan seminar on fuzzy sets and their applications:

Fuzzy set theorists handle a concept like "x is a tall man" by introducing a function h: $[0,3] \to [0,1]$ which gives a monotonic function from height (in metres, in this case) to $[0,1]$ – with $h(x) = 1$ for $x \geqslant 2$, $h(x) = 0$ for $x \leqslant 1$. So–to avoid the paradox of a threshold value, they happily accept the assignment of a precise "degree of tallness" to every height. Now–as Kochen shows in his *Application of fuzzy sets in psychology*–people can certainly draw a "degree of tallness" curve if pushed to it. But this does not show that our concept of tallness has such a form. Rather, we can recognize the *unambiguous* use of a word *in context*, and then can respond to aberrant situations if we have to. If we are meeting someone, and we are told they are "tall and light-skinned", our criteria will be completely different if we are in a group of Laplanders or Swazi. The language of probability theory "at least one standard deviation in the indicated direction away from the mean *of the current sample*" seems far more appropriate than the language of fuzzy set theory.

A horrifying thought–what if Newton had rejected the concept of mass, and sought to base his theory on a "degree of heaviness" between 0 and 1 for each object in the universe ?

Unlike his earlier papers which used lattice theory and category theory to place Zadeh's fuzzy sets in a rich mathematical context, J. A. Goguen's *On fuzzy robot planning* is an almost perfect microcosm of what ails fuzzy set theory. He starts by noting that humans can use "vague" descriptions like "Go about ten blocks north till you see a drugstore at a stoplight, then turn rightish ... ". The paper contains no results, but later on Goguen says that he plans to implement such directions on a computer, and that the idea will be to use *fuzzy lengths*–functions L: $[0,300] \to [0,1]$ where $[0,300]$ is the set of path lengths considered and $[0,1]$ is the set of weights–and *fuzzy angles*–functions D: $[0,360] \to [0,1]$ which assign a weight to each angle. A compiler is to be designed to transform verbal descriptions into instructions embodying such fuzzy terms. Goguen does not say how he would do it, but Tanaka and Mizumoto in the paper *Fuzzy programs and their execution* actually carry out a similar idea (conceived independently), and program a computer representation of a car to follow a route specified in verbal terms.

But this approach misses the whole point. When I receive an instruction "Go about ten blocks north ...", I do not respond by assigning 0 to blocks 0 through 6, 0.1 to block 7, 0.2 to block 8, 0.6 to block 9, 1.0 to block 10 and so on. Rather, I pick some number n less than 10, and conclude that I can ignore drugstores until I have gone n blocks, but that if I see one thereafter, I will turn. If I go 15 blocks without success, I will backtrack. Again, "turn rightish" requires no fuzzy sets–it is simply shorthand for "take the turning *closest* to 90° to the right."

Perhaps the most distressing mistake of the fuzzy set theorists is to believe that a natural language like English is *imprecise*. The fact that many people use English badly is no proof of *inherent* imprecision. The proper use of English is exquisitely precise without being unduly wordy. If there is only one road at an intersection making an angle between 40° and 140° to the right, then the instruction "turn right" is precise; if there are two roads, at angles of 80° and 100° right, then "turn right" is imprecise. No subtlety of fuzzy set theory will then raise your chance of taking the correct turn above 0.5. But a use of English tailored to the context–"Take the harder of the two right turns"–will solve the problem. When a friend gives us ambiguous instructions, we do not rejoice at the inherent fuzziness of language; we rail at our friend's inability to give proper directions. He failed to anticipate, and correct for, natural misinterpretations of his instructions.

Back to Goguen's article. His second section makes the point that, in language understanding, semantic cues may often predominate over syntactic. This is true. What is offensive is that Goguen implies that this is an insight of fuzzy set theory, and that workers in AI (Artificial Intelligence) have missed this point completely. First, there is nothing in Goguen's account that suggests that fuzzy set theory can offer a better framework than the frames or semantic networks of AI for representing semantic information. Second, he ignores the fact that many papers (three in the present volume) have argued for the use of fuzzy grammars in syntax–suggesting, at best, that fuzzy set theory is neutral on the relative merits of syntax and semantics. Third, and most importantly, a key feature of nearly *all* current AI work on language understanding [3] focuses on the proper use of semantics. And AI has already made contributions to the questions of hierarchical structuring and backtracking that Goguen calls for elsewhere in his paper.

Again and again in this collection, we see papers urging the merits of fuzzy set theory without any evidence that their approach is better than other, more established, approaches. For example, DePalma and Yau modify fuzzy grammars–in a way that removes them from the domain of fuzzy set theory, incidentally !–and apply them to recognition of handwriting. But the task on which they show moderate success is recognition of *presegmented* letters *e*, *i*, *l* and *t*–hardly the state of the art in a field where segmentation is recognized as the dominant challenge, and where the alphabet has 26 letters.

There are several papers on decision-making. All urge fuzzy sets as the way

to advance the field–but we never learn why this would help. Take, for example, the controversy [4]–not mentioned in these books–over the Club of Rome sponsored *The limits to growth*. How could fuzzy sets answer the criticism ? The key question of "What are the key relationships ? What is the coarsest meaningful level of aggregation ? How stable are the conclusions under perturbations of data ? Has proper use been made of available econometric findings ?" are not touched by this set of papers. And when it comes to analyzing the dependence of decisions on uncertain data, techniques of interval analysis and sensitivity analysis may well predominate. Interestingly, perhaps the only paper which shows a real awareness of the decision theory literature, that by Kung and Fu on *An axiomatic approach to rational decision making in a fuzzy environment*–and one of the few papers to contain a theorem of genuine mathematical interest–turns out (after a formal obeisance in the introduction) not to use fuzzy set theory at all, but to be a chapter in the theory of topological semigroups.

The Kaufmann volume, subtitled *Fundamental theoretical elements*, is the most depressing of the three. It is carefully written at a level that any undergraduate with basic mathematical training could understand; it has lots of examples–and yet it seems to me devoid of mathematical interest, whether pure or applied. Chapter 1 introduces the basic definitions; Chapter 2 looks at fuzzy graphs and fuzzy relations; Chapter 3 presents fuzzy switching theory; Chapter 4 presents fuzzy groupoids; while Chapter 5 generalizes to the case $\chi_A: X \to M$, where M has some suitable structure, e.g. that of a lattice. In all but Chapter 3, it seems only mildly unfair to caricature the method as "Take some simple concept, introduce graded membership where possible, and prove the obvious generalization of the theorem." Yet one is hardly encouraged by the exception. Chapter 3 concentrates on the construction of switching circuits that yield fuzzy functions whose values lie in some prespecified interval–yet the introduction to the chapter makes the grandiose claim: 'When software constructed with respect to a fuzzy logic becomes operational and when fuzzy hardware becomes industrially possible, then man-machine communication will be much more convenient, rapid, and better adapted to the solution of problems.' One sees a powerful delusional structure at work: "The human world is full of ambiguities. We thus need a fuzzy mathematics. So if we look at switching networks using fuzzy sets, we are en route to a humane and successful technology." There is no excuse for such sloppy thinking.

The book by Negoiţă and Ralescu is altogether superior, and is probably the best starting place for mathematicians who want to assess fuzzy set theory for themselves. They have realized that category theory provides the proper language for looking at generalized sets [5]; each chapter comes with useful historical and bibliographical remarks; and they cover twice as much material as Kaufmann in half the space. Yet the book is still disturbingly deficient. The category theory is but partially assimilated, so that many special cases of categorical results receive piecemeal treatment. No mention at all is made of

Topoi [6]. Even worse, the presentation of fuzzy systems in Chapter 4 provides ad hoc treatments of reachability and observability of limited applicability when papers cited in that chapter contain the proper general theory. The treatment of fuzzy languages in Chapter 5 ignores Schützenberger's elegant theory of formal languages [7] with coefficients in a semiring–which subsumed the concepts of Zadeh and Lee's [8] work on fuzzy grammars long before it was conceived.

The analysis of the chapters and papers in these three books could continue, but the above samples seem to yield an accurate picture. There is little here of purely mathematical interest–and what there is seems more likely to flourish under the appropriate banners of category theory or topological semigroups than under the catch-all of "fuzzy-set theory". Many contributions are simply exercises in generalization–and so the genuinely interesting results get lost amidst the chorus of "handle-cranking". Much of the work is philosophically naive–accepting sets with graded membership as an expression of language's imprecision, rather than seeking to understand how language can avail itself of context to combine precision with brevity. The applications suffer from being contributions to fuzzy set theory–and thus subject to no firm criteria–rather than contributions to decision theory, or pattern recognition, or whatever, and thus forced to meet–or change !–some well-established standards.

These last observations bring us once again to one of the most disturbing features of the literature collected within these volumes–its insularity. Virtually no workers in the area make any attempt to compare Zadeh's concepts with those in the literature of many-valued logic. Many papers are written on fuzzy languages or automata by authors ignorant of the work of Schützenberger. None of the three books under review makes any reference to such approaches to "fuzziness" as the interval analysis of Ramon Moore [9], or the lattice theory of computation of Dana Scott [10]. The "pure" papers rarely seem to have any depth to them–a familiar area is "fuzzified" with more or less unsurprising results. The applied papers use the language (though not always the theory) of fuzzy sets to tackle some problem–but never provide a hard-headed comparative study to show that their method is better than any others. Decision theory provides tools for combining judgmental weights and probabilities. I had hoped these volumes would assess those tools and show how fuzzy set theory could augment or replace them.

Let me close by stressing that this is a review of three books, and that I have made no attempt to sample the 600 items in the latest bibliography of research on fuzzy set theory. Respected colleagues who are adepts in the theory assure me that–had I not confined my reading of fuzzy set theory to the 1100 pages under review here–many of my questions would have been answered. I am happy to concede that the fuzzy set literature does contain some good papers, but still maintain that much remains to be done to winnow the chaff from the grain.

Since further books on fuzzy set theory are unavoidable, we may at least ask them to show a greater sensitivity to the relevant diverse sources of literature, and provide a comparative analysis which shows when and where the language of fuzzy set theory helps, and where it only adds fuzziness to the theory without in any way smoothing the original problem.

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Etude géométrique des espaces vectoriels, une introduction, by Jacques Bair and René Fourneau, Lecture Notes in Mathematics, no. 489, Springer-Verlag, Berlin, Heidelberg, New York, 1975, vii + 184 pp., $8.20.

In the axiomatic study of linear topological spaces over the real field, which flourished forty to twenty years ago, it soon became clear that the absolutely minimal requirement for a topology in a linear space L is that each line in L carry a copy of much of the structure of R. This finds expression in two aspects of segments, first, that each open segment in R is a neighborhood of all its points–an aspect that can usefully be generalized to linear spaces over all topological fields, and, second, that the two endpoints of each segment are accessible from the interior of the segment–an aspect which generalizes to linear spaces over ordered fields. These two attitudes lead to analogues of interior of a set in L and of derived set of a set in L.

The first attitude leads to a definition: x is called a core point of a subset A of L if for each line l through x the subset $l \cap A$ contains an open interval (in l) which contains x. Two topologies in L are suggested: For T, the neighborhoods of x are all the subsets of L which have x as a core point; for T_n, the neighborhoods of x are all the *convex* subsets of L which have x as a core point. T is not badly related to the linear operations in L; translation by an