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Techniques of multivariate calculation, by Roger H. Farrell, Lecture Notes in Math., Springer-Verlag, New York, x + 337 pp., $12.30.

This book is primarily concerned with the mathematical techniques useful in calculating the distribution of functions of random matrices X: $n \times p$ where X has a multivariate normal distribution. As motivation for both this review and much of the material in Farrell's book, I will begin by posing a problem and discussing three possible approaches to solving it. Suppose X is an $n \times p$ random matrix ($n \geqslant p$) and X has a density $f(x)$ with respect to Lebesgue measure, l, on the linear space of $n \times p$ matrices. Let $S = X'X \equiv \tau(X) \in \mathbb{S}_p$, where $\mathbb{S}_p$ is the set of all $p \times p$ nonnegative definite matrices (S is positive definite a.e.). The problem is to find the density function of S.

APPROACH 1. Assume that the density $f(X)$ is a function of S as is the case when the elements of X are independent and normal with mean 0 and variance 1. Then $f(X) = g(X'X)$ for some function g. Hence, the density of S is $g(S)$ with respect to the measure $\mu \equiv l \circ \tau^{-1}$ on $\mathbb{S}_p$. All that remains is to calculate the measure μ. Wishart did this in 1928 using a geometric argument which led to the density bearing his name (in the normal case). Of course, $\mu(dS) = c|S|^{(n-p-1)/2}dS$ where c is a constant.

When f is not a function of $X'X$, then the above argument is not available. Two alternative approaches which can be used are now considered.

APPROACH 2. The group $\mathcal{O}(n)$ of $n \times n$ orthogonal matrices acts on the left of X by $X \to \Gamma X$, $\Gamma \in \mathcal{O}(n)$. A maximal invariant function under this action is $\tau(X) = X'X = S$. The density of S with respect to the measure μ given above is q where $q(\tau(x)) = \int f(\Gamma x)\nu(d\Gamma)$. Here, ν is the invariant probability measure on $\mathcal{O}(n)$. This result was used by James (1954) to derive an integral expression for the density function of the noncentral Wishart distribution in the rank 3 case. A result similar to the one above for general compact groups is due to Stein and will be discussed subsequently.

APPROACH 3. Let G_p denote the group of $p \times p$ upper triangular matrices with positive diagonal elements. Also, let $V_{n,p}$ be the set of $n \times p$ matrices ψ which satisfy $\psi'\psi = I_p$. $V_{n,p}$ is called the Stiefel manifold. Each X which has rank p (those with rank less than p have Lebesgue measure 0) can be uniquely written as $X = \psi U$ with $\psi \in V_{n,p}$ and $U \in G_p$. Since $S = X'X = U'U$, a method for finding the density of S is to first find the joint density of ψ and U and then "integrate out" ψ to yield the marginal density of U. With the density of U at hand, the derivation of the density of S is rather routine since the Jacobian of the map $S \leftrightarrow U$ ($S = U'U$) is easily calculated. To obtain the joint density of ψ and U, note that $f(X) = f(\psi U)$ is the density of (ψ, U) with respect to whatever Lebesgue measure becomes in $V_{n,p} \times G_p$ under the transformation $X \to (\psi, U)$. An important result, usually proved via a Haar measure argument, is that the measure $l(dX)/|X'X|^{n/2}$ becomes $\mu_1(d\psi)\mu_2(dU)$ on $V_{n,p} \times G_p$ where: μ_1 is the unique invariant (under $\mathcal{O}(n)$ acting on $V_{n,p}$) probability measure on $V_{n,p}$ and μ_2 is a right invariant measure on G_p. Thus, the marginal density of U with respect to μ_2 is $|U'U|^{n/2}\int f(\psi(U)\mu_1(d\psi))$. This approach immediately yields the stochastic independence of ψ and U in the case $f(X) = g(X'X)$ considered in approach 1.

The above examples illustrate the type of problem and the mathematical techniques which arise in calculating multivariate distributions. For a thorough understanding of these examples, familiarity with Haar measure, transformation groups, matrix factorizations, and invariant differential forms is essential. Basically, it is this background material together with a large collection of examples that Farrell provides in these notes. The wide range of topics coupled with space limitations have necessarily restricted the amount of detail and number of proofs given. For example, in a short chapter on locally compact groups, the author discusses the existence and uniqueness of invariant measures on groups and quotient spaces, a useful result on factorization of measures, modular functions, cross sections, and the relationship between solvability and the Hunt-Stein condition.

In multivariate analysis, sample spaces are most often spaces of matrices which are also smooth manifolds (up to a null set). Further, the natural groups of transformations on these spaces are matrix groups. Therefore, to solve multivariate distribution problems, one needs to be able to manipulate and evaluate integrals with respect to differential forms over matrix spaces. Farrell's treatment of these topics begins with a discussion of exterior differential forms on manifolds. After calculating the explicit invariant measures on a variety of manifolds (lower triangular matrices, the orthogonal group, the Grassman and Stiefel manifolds), the author presents some useful decomposition results for $n \times p$ matrices. Applications of these results include a derivation of the density function for the sample canonical correlation coefficients and for the eigenvalues of a Wishart matrix.

The idea of finding densities of maximal invariants by integrating densities with respect to a Haar measure is due to Stein. In brief, the situation is the following: a locally compact group G operates measurably on $\mathfrak{X}$ and μ is a left invariant measure on $\mathfrak{X}$. A function τ: $\mathfrak{X} \to \mathfrak{Y}$ is maximal invariant under the action of G and f is a density on $\mathfrak{X}$ with respect to μ. If $X \in \mathfrak{X}$ has f as its density, the problem is to find the density function of $T \equiv \tau(X) \in \mathfrak{Y}$ (with respect to some measure which must be found– see Approach 2). If G is compact and ν is the invariant probability measure on G, then the answer is simple: namely, the density of T with respect to $\delta \equiv \mu \circ \tau^{-1}$ on $\mathfrak{Y}$ is $q(\cdot)$ where $q(\tau(x)) = \int f(gx)\nu(dg)$. An early application of this to the noncentral Wishart case where $G = \mathcal{O}(n)$ is due to James.

When the group G is not compact, the situation is much more complicated. The problem of finding a "nice" representation for a maximal invariant and the factorization of the measure μ are intimately connected and basic to providing a reasonable solution to the original question. One approach to the problem, developed by Wijsman, uses Lie group theory and the notion of cross sections. Schwartz has developed an alternative approach which is much closer to Stein's original approach. Farrell provides a detailed account of Schwartz's results (heretofore unpublished) and effectively uses the general multivariate linear model to motivate much of the theory. This section of the book provides the first teachable (not to mention readable) version of the theory that this reviewer has come across.

James' agrument (see Approach 2) shows that the problem of finding the density function of the noncentral Wishart distribution reduces to evaluating an integral of the form $\psi(A) \equiv \int \exp[\text{tr}\, \Gamma A]\nu(d\Gamma)$ where A is an $n \times n$ real matrix, $\Gamma \in \mathcal{O}(n)$ and ν is the invariant probability measure on $\mathcal{O}(n)$. When A has rank 1. the evaluation is not difficult and the answer is expressed as an infinite series in the one nonzero eigenvalue of AA'. T. W. Anderson derived a power series expansion for $\psi(A)$ when A has rank 2 and James did the same for rank 3 A's. The complexity of these power series increases drastically as the rank of A increases. Over the past twenty years, much theory has evolved in relation to zonal polynomial expansions of both $\psi(A)$ and more complicated integrals over $\mathcal{O}(n)$. After presenting a large body of algebraic theory, Farrell provides three different definitions of the zonal polynomials of a real symmetric matrix, plus some results related to zonal polynomial expansions of functions of a matrix argument Certainly, the theory leading to zonal polynomial expansions is very deep and beautiful mathematics. However, it is the reviewer's opinion that the introduction of zonal polynomial expansions into multivariate analysis has yielded neither significant theoretical insight nor useful numerical results.

Aside from a brief chapter on transform methods and one concerned with "random variable techniques," Farrell's book is about deriving distributions of functions of a normal random matrix using invariance and the mathematics surrounding such considerations. As stated by the author, he assumes a great deal of his readers. A short list of prerequisites would include measure and function theory, invariant integrals, complex analysis, a solid course in modern algebra, and a knowledge of real manifold theory. Unfortunately, this puts Farrell's presentation out of the reach of many graduate students and research workers in multivariate analysis. However, calculating (or even describing) multivariate distributions is often not easy and invariance is by far the most useful tool currently available.

For the most part, the book is written in the "definition, theorem, proof" style, but enough connective tissue is present to keep the reader's head above water. Armed with a plethora of reference books, the diligent student of multivariate analysis can and should profit from a thorough study of Farrell's book. The author has made a useful and welcome contribution to distribution theory in multivariate analysis.

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