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BOOK REVIEWS

Knots and links, by Dale Rolfsen, Publish or Perish, Inc., Berkeley, California, 439 pp., $15.00

I have a friend whom I do not see very often these days. When we manage to get together, we talk for hours. The conversation is sometimes relaxed, sometimes animated and rarely formal. My friend, I admit, carries most of the conversational burden, telling me the latest gossip and retelling old stories we both know but enjoy. He doesn't tell me where he picks up his news, and credit for some exploits is undoubtedly attributed on occasion to the wrong people. Sometimes when he says something clever and no one else is mentioned, I figure his mind is the source. He's charming and I never fail to count the hours with him well spent.

So it is with this book. It is charming and the most enjoyable mathematics book I have ever read. It is also a scholarly disaster. Ideas and theorems are usually unreferenced, leaving the unsophisticated reader to either assume the author as progenitor or categorize the result as not important enough for attribution (neither alternative will please the individual who first proved the theorem or introduced the idea). Occasionally, the author has selected the second person to prove a theorem as his reference; this is even more reprehensible, as we know how much easier it is to prove a known theorem than to do it first. For example this is the case in his failure to cite Mazur [**6**], [**7**] in a discussion of the generalized Schönflies theorem. The effect of these shortcomings is considerable. Just two other examples (selected from many) may suffice to give a flavor of the casual style, and demonstrate the author's sloppy approach to this aspect of scholarship.

On p. 116 in the proof of the asphericity of knots in S^3, the author says, "By a theorem of Whitehead, then, . . . ". He does not say which Whitehead, much less which theorem or where to find it.

On p. 105, Remark 8 gives a certain reference, reproduced as follows in its entirety" . . . Martin Gardner's Mathematical Recreations column in the Scientific American."

These considerations aside, let us consider what knot theory is about. Classical knot theory is concerned with the study of embeddings, or placements of a circle in 3-space (or its one point compactification, S^3).

Popular generalizations of this (aCparently) simple situation are to placements of several circles (links) in 3-space, to placements of one or more circles in a 3-manifold, and to the embeddings of higher dimensional spheres in still higher dimensional spaces.

Suppose we assume (as the author does) that the embeddings in question are tame. This means that the embedded sphere, or knot is a subcomplex of some triangulation of the space in which it is embedded. In the classical case this simply means the knot is polygonal.

The fundamental problem in knot theory is that of distinguishing knots.

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Two knots are the "same" or more formally belong to the same knot type if there is a homeomorphism of the containing space which throws one knot onto the other. Thus what we really want is a characterization of knot type.

Consideration of this geometric problem leads to algebra through the fundamental group of the complement of the knot in the containing space.

Let us for simplicity restrict attention to the classical case of a tamely embedded circle in 3-space. Denote the knot by k.

It is a consequence of work of Papakyriakopoulos **[10]** that the higher homotopy groups of the complement of a knot in S^3 vanish, so $S^3 - k$ is a so-called $K(\pi, 1)$ space whose homotopy type is completely determined by $\pi_1(S^3 - k)$–the fundamental group–or what is referred to as the group of the knot. Thus as a knot type invariant this group is potentially powerful (but not quite powerful enough, as two different knot types may have complements of the same homotopy type, yet the knots may belong to different types e.g. the square knot and granny knot). In passing it may be noted that the homology groups of $S^3 - k$ are independent of k by Alexander duality. Specifically $H_1(S^3 - k)$ is infinite cyclic, $H_i(S^3 - k) = 0, i > 1$.

The passage to algebra via the fundamental group has a return trip. In fact it is the interplay between geometry and algebra which gives knot theory much of its appeal. One return to geometry from algebra is based on what might be considered the fundamental theorem of knot theory: If the group of a knot is infinite cyclic, then the knot is "unknotted" or "trivial" i.e. it bounds an embedded disc. The converse is also true but very simple. The fundamental theorem is not so simple and is a consequence of the "Dehn Lemma" which was not proved until 1957 **[10]** when Papakyriakopoulos introduced a beautiful and novel method of removing singularities. His proof was later simplified by Shapiro and Whitehead **[12]**. (The latter proof is given in Rolfsen's book without reference to them.)

The Dehn Lemma states, (roughly speaking) that a nonsingular curve which bounds a singular disc in a 3-manifold, also bounds a nonsingular (embedded) disc, provided the singularities of the original disc avoid the boundary curve. Proofs of this lemma thus must somehow remove singularities.

Assuming the lemma, and a knot k with group infinite cyclic, one may prove the fundamental theorem as follows. Note that a nonsingular curve l close to k and having linking number 0 with k is homologous to 0. Since such a curve is homologous to 0 and $H_1(S^3 - k)$ is isomorphic to $\pi_1(S^3 - k)$ (since π_1 is abelian) l is homotopic to 0. As l is homotopic to 0 in $S^3 - k$, l bounds some disc (whose interior may be assumed to avoid l). By the Dehn Lemma this disc may be chosen to be nonsingular; since l is close to k this disc may be expanded to one bounded by k. Thus the abelian character of the group of the knot implies the triviality of the knot.

There are other geometric conclusions to be drawn from algebraic hypotheses. One of the most beautiful and subtle of these is Stallings' Theorem **[13]**: If the commutator subgroup of a knot group is finitely generated then the complement of the knot is a fiber space over a circle.

This general result was not known in any special case (except the trivial one) until Stallings proved it. Rolfsen devotes a number of pages to exhibiting this fibering (pictorially) in the case of an overhand knot, (the simplest possible case).

Returning to the basic problem of distinguishing knot types, there are a variety of invariants one may define. In a seminal paper **[11]** Seifert discussed one of great importance; the genus. Since k is a 1-cycle of the first homology group of S^3, and this group is trivial, k bounds a 2-chain; this 2-chain may be assumed to be a nonsingular 2-manifold with boundary k. The smallest genus for any such surface with boundary k is called the genus of k. Of course if this genus is 0 the knot is trivial, in case it is not we have a positive integer invariant of k. In some cases we can compute the genus from the group alone (e.g. when the commutator subgroup is finitely generated it is free of rank twice the genus **[9]**) in other cases we cannot. It is, however, always theoretically possible to calculate the genus from a picture of the knot **[14]**.

The genus is connected with a variety of other geometric and algebraic knot invariants, through both equalities and inequalities.

A whole class of invariants are obtained by consideration of branched and unbranched (ordinary) covering spaces of $S^3 - k$. The most important of these are the cyclic coverings. They correspond to subgroups of the knot group which are kernels of the mappings of the knot group onto a cyclic group. Rolfsen devotes several chapters to these spaces and their invariants.

The homology groups of these covering spaces, being knot invariants are of interest, and the homology of the covering corresponding to the commutator subgroup $[\pi_1, \pi_1]$ is, historically at least, of much interest. Denote this covering by X. Since $\pi_1/[\pi_1, \pi_1]$ is infinite cyclic, the group of covering translations of X is infinite cyclic, and this action subjects the homology of X to an action induced by these covering translations. If we let Z denote the group of covering translations and J the integers, then $H_1(X)$ is a JZ module.

This module (now called the Alexander module) was studied by Alexander **[1]**, Seifert **[11]**, Fox **[3]** and others. A precise description of the module is most easily given by a matrix whose entries lie in JZ, whose columns correspond to generators, and whose rows correspond to relations among the generators. It turns out that this matrix is square, and its determinant (which annihilates every module element) is the famous Alexander polynomial. Two properties characterize this polynomial invariant of a knot; first they take the value 1 at 1, second they are symmetric.

The most important new development in knot theory appears to be the use of surgery. This is beautifully described and utilized by Rolfsen. The crucial idea in this approach is to remove a solid unknotted tube from S^3, then replace it ("sew it back") with a twist. If the twisting is not too violent the patient will survive, in the sense that the resulting 3-manifold will be again S^3. However, previously unknotted curves will become knotted! In fact, one proves that any knot (in fact any 3-manifold) may be realized by judicious selection of tubes and twists in the complement of an unknotted curve. This relatively new point of view makes the proofs of standard theorems in the subject more understandable and new results more easily attainable. It renders certain notions (such as covering spaces and their completions to branched coverings) quickly accessible.

It is difficult to do knot theory and avoid 3-manifold theory, and Rolfsen devotes a good deal of space to the latter. One particularly nifty exercise asks the reader to verify that surgery on a collection of unknotted and simply linked tori, performed by giving the ith torus a simple twist of a_i turns yields the lens space $L(P, Q)$ where P/Q has the continued fraction expression

$$a_1 - \frac{1}{a_2 -} \frac{1}{a_3 -} \cdots \frac{1}{a_{n-1} -} \frac{1}{a_n}.$$

Since this exercise is unreferenced one might assume Rolfsen devised this construction. In fact I do not believe he did, as it is referred to in an equivalent form in *Differential manifolds and quadratic forms*, by Hirzebruch.

The completion of a finite covering of $S^3 - k$ to a branched covering yields a 3-manifold. In fact, any 3-manifold is a branched cover of a link (Alexander [2]). Recently it has been shown by Hilden [4] and Montesinos [8] that any 3-manifold is a 3-fold branched covering of some knot.

Returning to the book itself, there are some good things to be said about it.

The pictures are marvelous and are usually accompanied by crystal clear and concise explanations.

The proofs, while convincing, generally contain no more detail than propriety requires, and a real attempt is made to provide the idea of the proof. In fact, in general the author excels at showing how things are done. It is clear he enjoys the process of communicating mathematical ideas.

Finally a question of taste; his judgement as to what is "remarkable", and mine do not coincide. That the range of groups which are the fundamental group of the complement of a tame n-sphere in the $n + 2$ sphere [5] is independent of n, for $n > 2$ does not seem particularly remarkable to me though it does to him. That the complement of even one nontrivial knot in the 3-sphere fibers over the circle seems remarkable to me but not to him. I find the latter result particularly surprising since it can be discovered geometrically and hence was accessible for at least 35 years before it was discovered, [13].

In the dim past there was an informal system of free distribution of lecture notes from courses given at various universities. These did not have the imprimature of a publishing house and the price was right, so criticism was muted to the point of silence. If now we are to pay ($15 in this case) for lecture notes, and if people are going to make a buck on such an enterprise, we are entitled to expect a good deal more care in their preparation. Misprints (of which there are several dozen), sloppy and awkward writing (of which there is a bit), and improper and lazy referencing (of which there is a super abundance) are ingredients of a rip off. This book would qualify for that characterization were it not for the brilliance of selectivity and obvious enthusiasm and skill of the author. The examples in the book are hard to find elsewhere, no one other book contains the quantity and timeliness of material it presents. I loved reading it, I learned a great deal from it and I recommend its purchase by anyone with even a passing interest in low dimensional geometric topology. Just be careful when it comes to using it for the purpose of kudology.

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LEE P. NEUWIRTH

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Ordinary differential equations in the complex domain, by Einar Hille, Pure and Applied Mathematics Series, John Wiley & Sons, New York, 1976, xi + 484 pp., $27.95.

The usual basic concepts and methods for ordinary differential equations in the complex domain are explained without going into tedious details. The reviewer believes that the readers will be able to familiarize themselves with those basics and that this book will be appreciated very much. It is fair to say that the interest of the author is more focussed on "Method" than "Intrinsic Meaning".

Through reading, the impression was that of listening to "Grandfather" while strolling with him in a quiet cemetary. He talks about good old days and beautiful people. In this book Lappo-Danilevskij is still alive, but Grothendieck does not exist. The introduction has two parts: Part I is "Algebraic and Geometric Structures" and Part II is "Analytical Structures". The contents of Part I actually belong to "Functional Analysis". They are not algebro-geometric in the sense of Grothendieck-Deligne-Katz (P. Deligne [3]). "Analytical Structures" means a collection of traditional basics for functions of one complex variable. The concept of analytic continuation is explained, but Riemann surfaces are not clearly defined. The resources look very meager. What can be accomplished? Indeed, not very much more than