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SEMIAMARTS AND FINITE VALUES

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Let X_n be a sequence of real-valued random variables adapted to an increasing sequence of σ-algebras F_n. We denote by T, T_f, $\overline{T}$ respectively the collection of bounded, finite, and arbitrary stopping times for $(F_n)_{n\in \mathbf{N}}$. This paper reports on recent progress concerning the theory of *semiamarts*, i.e. processes for which $(EX_\tau)_{\tau\in T}$ is bounded, initiated in [3], and the theory of *amarts*, i.e. processes for which $\lim_{\tau\in T} EX_\tau$ exists. We relate the notion of semiamart to processes of interest in the theory of optimal stopping (cf. [2]), namely X_n such that $|EX_\mu| < \infty$ for $\mu \in T_f$, or for $\mu \in \overline{T}$. For independent random variables X_n and for processes of the form $X_n = c_n^{-1} \Sigma_{i=1}^n Y_i$ with increasing c_n's and independent nonnegative Y_i's, a new dominated estimate

$$E(\sup X_n^+) \leqslant K \sup_{\mu\in\overline{T}} EX_\mu \quad (=KV(\overline{T}))$$

with $K = 2$ in the first and $K < 5.46$ in the second case, shows that such processes are semiamarts if and only if $\sup|X_n|$ is integrable. Also in the case when $F_n = F_m$ for all $n, m \in \mathbf{N}$, a semiamart has a necessarily integrable supremum. This observation is used to construct averages of aperiodic stationary sequences, which are not semiamarts—thereby strengthening a result announced by A. Bellow [1]. This can be done also in the "descending" case, i.e. when the time domain $\mathbf{N}$ is replaced by $-\mathbf{N}$ (see [3]); thus our results indicate that there are no connections between the amart theory and the ergodic theory of point transformations.

THEOREM 1 (RIESZ DECOMPOSITION FOR SEMIAMARTS). *Every semiamart (X_n, F_n) can be represented as $X_n = Y_n + Z_n$ where (Y_n, F_n) is a martingale and (Z_n, F_n) is an L_1-bounded semiamart such that for each $A \in \bigcup F_m$*

$$\liminf_n \frac{1}{n}\sum_{i=1}^n \int_A Z_i \leqslant 0 \leqslant \limsup_n \frac{1}{n}\sum_{i=1}^n \int_A Z_i.$$

This generalizes the Riesz decomposition for amarts [3]. A variant of Theorem 1 permits us to give necessary and sufficient conditions for the uniqueness of the Riesz decomposition. One consequence of the Riesz decomposition is:

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THEOREM 2. *Let X_n be a semiamart (amart) such that for some $\alpha \geqslant 1$ $\Sigma_{i=1}^{\infty} i^{-(1+\alpha)} E|X_i - X_{i-1}|^{2\alpha} < \infty$; then $\sup|X_n|/n < \infty$ a.s. (resp. $X_n/n \longrightarrow 0$ a.s.).*

Theorem 2 extends the strong law of large numbers for martingale differences; a somewhat weaker version of this, and of the next theorem, appears in [3].

THEOREM 3 (AMART OPTIONAL SAMPLING THEOREM). *Let $\mu_n \in T_f$, $\mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \cdots$. Let X_n be an amart, $\hat{X}_n = X_{\mu_n}$ and assume*

(a) $E|\hat{X}_n| < \infty \; \forall n \in \mathbf{N}$ *and*

(b) $\lim_{N \longrightarrow \infty} \int_{\{\mu_n > N\}} |X_N| = 0 \; \forall n \in \mathbf{N}$.

Then $(\hat{X}_n, \mathcal{G}_n)$ is an amart where $\mathcal{G}_n = \mathcal{F}_{\mu_n} = \{A \in \mathcal{F}: A \cap \{\mu_n = k\} \in \mathcal{F}_k \forall k\}$. If also $\mu_n \longrightarrow \infty$ then the Riesz decomposition of $\hat{X}_n$ has the martingale part $Y_n = Y_{\mu_n}$ and the potential part $\hat{Z}_n = Z_{\mu_n}$, where $Y_n + Z_n$ is the amart Riesz decomposition of X_n.

THEOREM 4. *There exists a semiamart which converges a.s. and in L_1 but is not an amart.*

There exist two simple methods of construction of amarts and semiamarts: (1) each adapted sequence X_n is a semiamart if $\sup|X_n| \in L_1$. Such a sequence is an amart iff in addition X_n converges a.s.; (2) quasimartingales are amarts.

THEOREM 5. *In general a semiamart or amart cannot be decomposed into two summands arising from constructions (1) and (2). In fact, there exists a nonnegative predictable amart which is a potential (the martingale part in its Riesz decomposition vanishes), with $\sup_n E(X_n \log^+ X_n) \leqslant 1$ and $E \sup X_n = \infty$.*

THEOREM 6. *Let $(X_n)_{n \in \mathbf{N}}$ be a sequence of adapted random variables for the increasing sequence $(\mathcal{F}_n)_{n \in \mathbf{N}}$, with $\sup E|X_n| = M < \infty$. (X_n) is a semiamart iff for each $\nu \in \overline{T}$ such that $E(1_{\{\nu<\infty\}} X_\nu)$ is defined as an extended real number, one has $|E(1_{\{\nu<\infty\}} X_\nu)| < \infty$. If the σ-algebra $\mathcal{F}_\infty$ generated by all $\mathcal{F}_n$'s is nonatomic, a further equivalent condition is: for each $\nu \in T_f$ such that EX_ν is defined as an extended real number, one has $|EX_\nu| < \infty$.*

If $(EX_\tau)_{\tau \in T}$ is unbounded from above one can find ν with $EX_\nu^- < \infty$ and $EX_\nu^+ = \infty$. Thus the theorem can be interpreted as saying that for L_1-bounded processes with infinite value $V(T) = \sup_{\tau \in T} EX_\tau$, the value $V(\overline{T})$ is assumed, and $V(T_f)$ is assumed if $\mathcal{F}_\infty$ is nonatomic. In the descending case $V(T_f) = \infty$ is assumed if (X_n) is L_1-bounded *or* each $\mathcal{F}_{-n}$ is nonatomic. Since then $V(T_f) = \infty$ is equivalent to $V(T) = \infty$, this yields an analogous characterization of descending semiamarts.

THEOREM 7. *If (X_n) is adapted to $(\mathcal{F}_n)$ and X_{n+1} is independent of $\mathcal{F}_n$ for all n, then $E \sup X_n^+ \leqslant 2V(\overline{T})$.*

We only showed the existence of a constant K_0 such that $2 \leqslant K_0 \leqslant 4$, and $E \sup X_n^+ \leqslant K_0 V(\overline{T})$. That K_0 may be chosen equal to 2 is due to D. Garling.

Now let (Y_n) be adapted to the increasing family (F_n) and assume that Y_{n+1} is independent of F_n for all n. Call (X_n) a *sequence of averages of nonnegative independent random variables* if X_n is of the form $X_n = c_n^{-1} \Sigma_{i=1}^n Y_i$ with $1 \leqslant c_1 < c_2 < \cdots$.

THEOREM 8. *If (X_n) is a sequence of averages of nonnegative independent random variables then $E(\sup X_n) < 5.46$ where $V = V(\overline{T}) = V(T_f) = V(T)$.*

This result has an interesting probabilistic interpretation. If X_n is the fortune of a player at time n, then V is the maximal expected gain of a player A using nonanticipating stopping rules. $E \sup X_n$ equals $\sup_\mu EX_\mu$ where the supremum is over *all* measurable random variables $\mu: \Omega \longrightarrow \mathbf{N}$. Thus $E \sup X_n$ is the maximal expected gain of a player B endowed with complete foresight. The theorem may be interpreted as saying that, whatever be the sequence of distributions, the odds 5.46:1 are favorable to A even against an omniscient opponent B playing the same game.

A consequence of Theorem 9 is that a sequence of averages of nonnegative independent random variables is a semiamart for (F_n) iff $\sup X_n \in L_1$.

Call a point-transformation S *aperiodic* if there exists no measurable B with $P(B) > 0$ such that for some $n \in \mathbf{N}$ and all measurable $A \subset B$, the symmetric differences $A \,\Delta\, S^{-n}A$ has measure 0. The result of A. Bellow [1] is strengthened by

THEOREM 9. *If S is an aperiodic invertible measure preserving transformation of (Ω, F, P) then there exists an $f \in L_1^+$ for which $X_n = n^{-1} \Sigma_{k=0}^{n-1} f \circ S^k$ is not a semiamart, ascending or descending.*

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