BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, July 1977

ULTRAFILTERS: SOME OLD AND SOME NEW RESULTS

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I am grateful to Peter Freyd for his generous introductory comments, and I am grateful also to the [Selection] Committee for extending the invitation to speak to you.

Last month a colleague with whom I attempted to discuss today's remarks suggested drily that as half of the team responsible for [CN2] I have probably already said more than enough about ultrafilters, and that if I insist on pursuing the matter further today I could do so most gracefully and efficiently simply by offering a complimentary copy of [CN2] to each of you. Eschewing that advice I shall in the hour allotted to me attempt to achieve the following three goals.

(A) To acquaint you with what I think are some of the most basic, fundamental facts about ultrafilters on a discrete topological space; this material is sufficiently simple and elegant that it can be absorbed comfortably into a first-year graduate course in general topology.

(B) To give some partial results, less definitive and less conclusive than the optimal theorems available, concerning the existence of particular ultrafilters with special properties; I hope that the results chosen in this connection have the complementary virtues that they are sufficiently powerful to handle most of the situations treated by the more powerful results which we shall ignore, and that their proofs are significantly simpler than those of the more general results.

(C) To record some results about ultrafilters which came to my attention after the publication of [CN2]; I have chosen today to emphasize three relatively new results which are not formally concerned with ultrafilters and which indeed make no mention of ultrafilters in their statements, but which nevertheless have been given proofs in which ultrafilters play an important catalytic role.

My hope is that (even) those of you not professionally inclined toward topology or set theory will find something potentially useful, or amusing, among the basic results given in (A). The theorems selected for inclusion in (B) are given not only because of the intrinsic beauty and elegance of their proofs, but also because they serve to indicate the principal sorts of questions

An address delivered at the Eastern Sectional Meeting of the American Mathematical Society in New York, New York on April 13, 1976; received by the editors August 15, 1976.

AMS (MOS) subject classifications (1970). Primary 04A20, 04A30, 10A45, 54A25, 54C45, 54D20, 54G05.

[1] The author gratefully acknowledges partial support received from the National Science Foundation under grants MPS73-08937A02 and MCS76-05821.

asked and answered recently by several leading workers in the theory of ultrafilters. (The recent applications of ultrafilters to model theory, and the good ultrafilters of Keisler, however, will not be discussed.) I hope, finally, that even the experts among you may find something new or interesting in (C). I do not believe that the three proofs given there have yet been published, but in each case the discovering mathematician has authorized or encouraged its inclusion in this talk. It is anticipated that similar proofs of these and perhaps of related results may appear in **[C3]**.

Whatever I shall say today is influenced strongly by **[CN2]** and hence, indirectly but nevertheless profoundly, by my colleague and coauthor S. Negrepontis.

A. Some Fundamental Facts About Ultrafilters

1. Notation and definitions. Throughout these remarks, by a *space* we shall mean a completely regular Hausdorff space. For spaces X and Y we denote by $C(X, Y)$ the set of continuous functions from X into Y, and we write

$$\mathcal{Z}(X) = \{f^{-1}(\{0\}): f \in C(X, [0, 1])\}.$$

We shall be concerned principally with discrete spaces. We do not distinguish notationally between a cardinal number α and the discrete space whose underlying set of points is that cardinal; the symbol ω denotes both the least infinite cardinal and the countably infinite discrete space. The symbols ξ, η and the like denote ordinals, so for α a cardinal we have

$$\alpha = \{\xi: \xi \in \alpha\} = \{\xi: \xi < \alpha\}.$$

If X is a set we denote by $\mathcal{P}(X)$ the power set of X; that is,

$$\mathcal{P}(X) = \{A: A \subset X\}.$$

For every space X there are enough continuous functions from X into $[0, 1]$ to distinguish points and to separate points from closed sets, and consequently (see for example **[Kl]** or **[En]**) the embedding $e: X \to P = \prod_{f \in \mathcal{F}} [0, 1]_f$ (where $\mathcal{F} = C(X, [0, 1])$ and $[0, 1]_f = [0, 1]$ for each $f \in \mathcal{F}$) is a topological embedding (*i.e.*, a homeomorphism into). Since each $f \in \mathcal{F}$ is essentially a projection from $e[X]$ into $[0, 1]_f$, each such f extends continuously over P and, hence, over the intermediate space $\mathrm{cl}_P e[X]$. The space $\mathrm{cl}_P e[X]$ is denoted βX and called the *Stone-Čech compactification* of X. If $\beta Y \subset Q = \prod_{g \in \mathcal{G}} [0, 1]_g$ with $\mathcal{G} = C(Y, [0, 1])$, and if $f \in C(X, Y)$, then $g \circ f \in C(X, [0, 1])$ for each $g \in \mathcal{G}$ and hence $g \circ f$ extends continuously over βX. The product function takes X into βY and the following basic result of Čech **[Č]** is now available (see also **[GJ**, Chapter 11**]** and **[CN3]** for proofs).

1.1. Theorem. *For every space X the space βX satisfies*

(a) *every continuous function from X into a compact space extends continuously over βX*; *and*

(b) *βX is, up to a homeomorphism leaving X fixed pointwise, the only compact* *space in which X is dense to which every continuous function from X into* $[0, 1]$ *extends continuously.*

For $f \in C(X, Y)$ with Y compact, we denote by $\bar{f}$ that (unique) element g of $C(\beta X, Y)$ such that $f \subset g$; the function $\bar{f}$ is called the *Stone extension* of f.

An alternative definition of βX, closer in spirit to the Boolean-algebra approach of M. H. Stone [**St**], departs from the family $\mathfrak{Z}(X)$. As a set, βX is taken to be the set of maximal filters of elements of $\mathfrak{Z}(X)$–*i.e.*, the z-ultrafilters topologized so that $\{\{p \in \beta X\colon Z \in p\}\colon Z \in \mathfrak{Z}(X)\}$ is a basis for the closed subsets of βX; in this approach the inclusion $X \to \beta X$ is achieved by identifying an element p of X with the z-ultrafilter $\{Z \in \mathfrak{Z}(X)\colon Z \in p\}$. The familiar proof that βX, so defined, is a compact space with property (a) (and hence also (b)) of Theorem 1.1 is given, for example, in [**GJ**, Theorem 6.5] and [**CN2**, Theorem 2.6].

For every subset A of a discrete space α we have $A \in \mathfrak{Z}(\alpha)$, so the topology of $\beta(\alpha)$ is given by the following particularly tractable identity:

$$p \in \mathrm{cl}_{\beta(\alpha)} A \text{ if and only if } A \in p \qquad (A \subset \alpha, p \in \beta(\alpha)).$$

We note for $A \subset \alpha$ that $\mathrm{cl}_{\beta(\alpha)} A \cap \mathrm{cl}_{\beta(\alpha)} (\alpha \backslash A) = \varnothing$; indeed otherwise there is $p \in \beta(\alpha)$ such that $p \in \mathrm{cl}_{\beta(\alpha)} A \cap \mathrm{cl}_{\beta(\alpha)} (\alpha \backslash A)$, so that $\varnothing = A \cap (\alpha \backslash A) \in p$, a contradiction. Since α is dense in $\beta(\alpha)$ we have

$$\beta(\alpha) = \mathrm{cl}_{\beta(\alpha)} A \cup \mathrm{cl}_{\beta(\alpha)} (\alpha \backslash A)$$

for every $A \subset \alpha$. It follows that $\mathrm{cl}_{\beta(\alpha)} A$ is open and closed in $\beta(\alpha)$ and the following simple result is available.

1.2. LEMMA. *Let $p \in \beta(\alpha)$. Then $\{\mathrm{cl}_{\beta(\alpha)} A\colon A \in p\}$ is a neighborhood base for $\beta(\alpha)$ at p.*

PROOF. If U is an open neighborhood of p in $\beta(\alpha)$ then p is not an element of the closed set $\beta(\alpha) \backslash U$, and according to the definition of the topology of $\beta(\alpha)$ there is $B \subset \alpha$ such that $\mathrm{cl}_{\beta(\alpha)} B \supset \beta(\alpha) \backslash U$ and $p \notin \mathrm{cl}_{\beta(\alpha)} B$. We define $A = \alpha \backslash B$ and we have, from the remarks above, that $p \in \mathrm{cl}_{\beta(\alpha)} A$ and $\mathrm{cl}_{\beta(\alpha)} A \subset U$.

The elements p of $\beta(\alpha)$ such that $p \in \alpha$ are *principal* ultrafilters (or *fixed* [**GJ**]); the elements of $\beta(\alpha) \backslash \alpha$ are *nonprincipal* (or *free* [**GJ**]). We say that p is *uniform* on α if $|A| = \alpha$ for every $A \in p$. The set of uniform ultrafilters on α is denoted $U(\alpha)$.

For $\alpha \geqslant \omega$ and $A \subset \alpha$ we set

$$\hat{A} = (\mathrm{cl}_{\beta(\alpha)} A) \cap U(\alpha).$$

We note the following consequence of Lemma 1.2.

1.3. COROLLARY. *Let $\alpha \geqslant \omega$ and $p \in U(\alpha)$. Then $\{\hat{A}\colon A \in p\}$ is a local basis of $U(\alpha)$ at p.*

The following statements are clear for every cardinal $\alpha > 0$: $\varnothing \neq \alpha \subset \beta(\alpha)$; $U(\alpha) \subset \beta(\alpha)\backslash\alpha$; and $\alpha = \beta(\alpha)$ if $\alpha < \omega$. We note also that $U(\alpha)$ is a closed subset of $\beta(\alpha)\backslash\alpha$ (and hence compact). Indeed if $p \in (\beta(\alpha)\backslash\alpha)\backslash U(\alpha)$ then there is $A \in p$ such that $|A| < \alpha$, and it follows from the remarks preceding Lemma 1.2 that $\mathrm{cl}_{\beta(\alpha)} A$ is a neighborhood of p in $\beta(\alpha)$ disjoint from $U(\alpha)$.

Ultrafilters were apparently first defined, and shown to exist (on ω), by F. Riesz **[Ri]** and Ulam **[U1]**, respectively. Today we recognize that the following fundamental existence theorem, without which a discussion of ultrafilters runs the risk of appearing potentially vacuous, is proved easily by a routine application of Zorn's lemma to the family $\{A \subset \alpha: |\alpha\backslash A| < \alpha\}$.

1.4. LEMMA. *Let $\alpha \geqslant \omega$. Then $U(\alpha) \neq \varnothing$.*

We note in passing that the (apparent) dependence on the Axiom of Choice of the two definitions of $\beta(\alpha)$ given above is not superficial and cannot be eliminated. It is known **[So]**, **[J**, Problem 5.24, p. 82] that, the Axiom of Choice not being assumed, it is consistent with the remaining axioms of Zermelo-Fraenkel set theory that every subset of $\mathbf{R}$ is Lebesgue-measurable; but Sierpiński **[Si]** has shown (see also Semadeni **[Se]**) that if $\beta(\omega)\backslash\omega \neq \varnothing$, then without the Axiom of Choice one may construct a nonmeasurable subset of $\mathbf{R}$. In this connection see also **[C1]**, **[Sl]**.

2. **Some cardinals associated with $\beta(\alpha)$.** It is easy to prove that if Y is a space and X is dense in Y, then $|Y| \leqslant 2^{2^{|X|}}$; indeed it follows from the Hausdorff separation property for Y that the function

$$y \to \{U \cap X: U \text{ is a neighborhood of } y\}$$

is a one-to-one function from Y into $\mathcal{P}(\mathcal{P}(X))$. Since α is dense in its Stone-Čech compactification $\beta(\alpha)$ we have $|\beta(\alpha)| \leqslant 2^{2^\alpha}$. The reverse inequality, contained essentially in a set-theoretic argument given by Fichtenholz and Kantorovitch **[FK]** and Hausdorff **[Hs]**, was first proved explicitly by Pospíšil **[P1]**. The proof in 2.4 below, based on the so-called Hewitt-Marczewski-Pondiczery theorem **[He]**, **[Ma]**, **[P]**, follows an argument suggested by Mrówka **[Mr]**.

The *density character* of X, denoted $\mathrm{d}X$ or $\mathrm{d}(X)$, is the least cardinal which is the cardinal number of a dense subset of X.

The following argument is a minor but nifty variation, shown to me recently by Teklehaimanot Retta, on the argument normally used.

2.1. THEOREM. *If $\alpha \geqslant \omega$, then* $\mathrm{d}(\alpha^{(2^\alpha)}) = \alpha$.

PROOF. Let $\mathfrak{B}$ denote the set of open-and-closed subsets of the compact space 2^α. It is clear that $\mathfrak{B}$ is a base for 2^α, that $|\mathfrak{B}| = \alpha$ (since $\alpha \geqslant \omega$), and that for every faithfully indexed finite subset $\{f_k: k < n\}$ of 2^α there is a partition $\{A_k: k < n\}$ of 2^α by pairwise disjoint elements of $\mathfrak{B}$ such that $f_k \in A_k$ for $k < n$.

For each (necessarily finite) partition $\mathcal{Q} = \{A_k: k < n\}$ of 2^α by elements of $\mathfrak{B}$ and each $\varphi \in \alpha^n$, we define $\Phi_{\mathcal{Q},\varphi} \in \alpha^{(2^\alpha)}$ by

$$\Phi_{\mathcal{Q},\varphi}(f) = \varphi(k) \quad \text{if } f \in A_k,$$

and we set

$$D = \{\Phi_{\mathcal{Q},\varphi}: \varphi \in \alpha^{|\mathcal{Q}|}\}.$$

It is clear that D is dense in $\alpha^{(2^\alpha)}$, and that

$$|D| \leqslant \sum \{\alpha^n \cdot |\mathfrak{B}|^n: n < \omega\} = \alpha.$$

The relation $\mathrm{d}(\alpha^{(2^\alpha)}) \geqslant \alpha$ is obvious.

Theorem 2.1, a special case of [P], is enough to prove the result known as the Hewitt-Marczewski-Pondiczery theorem.

2.2. THEOREM. *Let $\alpha \geqslant \omega$ and let $\{X_i: i \in I\}$ be a set of spaces such that $\mathrm{d}(X_i) \leqslant \alpha$ for each $i \in I$. If $|I| \leqslant 2^\alpha$, then $\mathrm{d}(\prod_{i\in I} X_i) \leqslant \alpha$.*

PROOF. For $i \in I$ there is a (continuous) function f_i from α onto a dense subset D_i of X_i. The product function f: $\alpha^I \to \prod_{i\in I} D_i$, defined by the rule $f(p)_i = f_i(p_i)$, takes α^I onto a subset of $\prod_{i\in I} D_i$ which is dense in $\prod_{i\in I} D_i$ and hence in $\prod_{i\in I} X_i$. From Theorem 2.1 there is a dense subset D of α^I such that $|D| \leqslant \alpha$; it is clear that $f[D]$ is dense in $\prod_{i\in I} X_i$.

2.3. LEMMA. *Let $\alpha \geqslant \omega$. There is a (continuous) function f from α into $\alpha^{(2^\alpha)}$ such that the Stone extension $\bar{f}$: $\beta(\alpha) \to (\beta(\alpha))^{2^\alpha}$ satisfies $\bar{f}[U(\alpha)] = (\beta(\alpha))^{2^\alpha}$.*

PROOF. Let $\langle A_\xi: \xi < \alpha\rangle$ be a decomposition of α into pairwise disjoint subsets of cardinality α, let $\langle p_\xi: \xi < \alpha\rangle$ be a dense subset of $\alpha^{(2^\alpha)}$ (from Theorem 2.3), and for $\xi < \alpha$ let q_ξ be an ultrafilter uniform over A_ξ (from Lemma 1.2). Define f: $\alpha \to \alpha^{(2^\alpha)}$ by the rule $f[A_\xi] = p_\xi$ and note from the continuity of $\bar{f}$ that $\bar{f}(q_\xi) = p_\xi$ for each $\xi < \alpha$. It follows that $\bar{f}[U(\alpha)]$ is a compact subset of $(\beta(\alpha))^{2^\alpha}$ containing the dense set $\langle p_\xi: \xi < \alpha\rangle$, so that $\bar{f}[U(\alpha)] = (\beta(\alpha))^{2^\alpha}$, as required.

2.4. COROLLARY. *If $\alpha \geqslant \omega$, then $|U(\alpha)| = |\beta(\alpha)\backslash\alpha| = |\beta(\alpha)| = 2^{2^\alpha}$.*

PROOF. We have noted already that $U(\alpha) \subset \beta(\alpha)\backslash\alpha \subset \beta(\alpha)$ and that $|\beta(\alpha)| \leqslant 2^{2^\alpha}$. The relation $|U(\alpha)| \geqslant 2^{2^\alpha}$ follows from Lemma 2.3 and the inequality $|\beta(\alpha)| > 1$ (a consequence of the inclusion $\beta(\alpha) \supset \alpha$).

For X a space the *weight* of X, denoted wX or w(X), is the least cardinal which is the cardinal number of a basis for X. If $p \in X$ we denote by $\chi(p, X)$ the *local weight* of X at p–*i.e.*, $\chi(p, X)$ is the least cardinal which is the cardinal number of a local basis of X at p; if $\alpha \geqslant \omega$ and $p \in U(\alpha)$, we denote $\chi(p, U(\alpha))$ simply by $\chi(p)$.

We indicate below that the last three weak inequalities in Theorem 2.5 are in fact equalities.

2.5. THEOREM. *Let $\alpha \geqslant \omega$ and $p \in U(\alpha)$. Then*

$$\alpha < \chi(p) \leqslant \mathrm{w}U(\alpha) \leqslant \mathrm{w}(\beta(\alpha)\backslash\alpha) \leqslant \mathrm{w}(\beta(\alpha)) \leqslant 2^{\alpha}.$$

PROOF. We show first that $\alpha < \chi(p)$. If $\chi(p) \leqslant \alpha$ then according to Corollary 1.3 there is $\{A_\xi : \xi < \alpha\} \subset \mathcal{P}(\alpha)$ such that for every neighborhood U of p in $U(\alpha)$ there is $\xi < \alpha$ such that $p \in \hat{A}_\xi \subset U$. We choose distinct elements p_0, q_0 of A_0 and then recursively, if $\xi < \alpha$ and p_η, q_η have been chosen for all $\eta < \xi$, we choose distinct elements p_ξ, q_ξ of the set

$$\hat{A}_\xi\backslash(\{p_\eta : \eta < \xi\} \cup \{q_\eta . \eta < \xi\});$$

such a choice is possible because $A_\xi \in p \in U(\alpha)$.

We define $P = \{p_\xi : \xi < \alpha\}$ and $Q = \{q_\xi : \xi < \alpha\}$. We have $p \in \mathrm{cl}_{\beta(\alpha)} P \cap \mathrm{cl}_{\beta(\alpha)} Q$ and hence $\varnothing = P \cap Q \in p$, a contradiction.

The relations $p \in U(\alpha) \subset \beta(\alpha)\backslash\alpha \subset \beta(\alpha)$ imply $\chi(p) \leqslant \mathrm{w}U(\alpha) \leqslant \mathrm{w}(\beta(\alpha)\backslash\alpha) \leqslant \mathrm{w}(\beta(\alpha))$, so it remains only to show $\mathrm{w}(\beta(\alpha)) \leqslant 2^\alpha$. We take $\mathcal{F} = C(\alpha, [0, 1])$ and we note that this follows from the relations $\beta(\alpha) \subset [0, 1]^{\mathcal{F}}$, $|\mathcal{F}| = 2^\alpha$ and $\mathrm{w}([0, 1]^{2^\alpha}) = 2^\alpha$; alternatively one may note that from Lemma 1.2 we have $\mathrm{w}(\beta(\alpha)) \leqslant |\mathcal{P}(\alpha)| = 2^\alpha$.

The following lemma and theorem, taken from Pospíšil **[P2]**, show that there are $p \in U(\alpha)$ such that $\chi(p) = 2^\alpha$; see also Juhász **[J1]**, **[J2]**.

2.6. LEMMA. *Let $\alpha \geqslant \omega$, let X and Y be compact spaces, and let f be a continuous function from X onto Y. Let $\{V_\xi : \xi < \alpha\}$ be a family of neighborhoods of $p \in Y$ such that*

$$\textit{if } I \in \mathcal{P}(\alpha) \textit{ and } |I| = \omega \textit{ then } \mathrm{int}\Big(\bigcap_{\xi \in I} V_\xi\Big) = \varnothing.$$

Then there is $q \in f^{-1}(\{p\})$ such that $\chi(q, X) \geqslant \alpha$.

PROOF. We set

$$U_I = \operatorname*{int}_X \Big(\bigcap_{\xi \in I} f^{-1}(V_\xi)\Big) \quad \text{for } I \in \mathcal{P}(\alpha).$$

We note that if U is open in X and $f^{-1}(\{p\}) \subset U$ then $p \notin f[X\backslash U]$ and hence $p \in \mathrm{int} f[U]$. Since

$$\begin{aligned}\mathrm{int} f[U_I] &= \mathrm{int} f\Big[\operatorname*{int}_X \bigcap_{\xi \in I} f^{-1}(V_\xi)\Big] \subset \mathrm{int} f\Big[\bigcap_{\xi \in I} f^{-1}(V_\xi)\Big] \\ &= \mathrm{int}\Big(\bigcap_{\xi \in I} V_\xi\Big) = \varnothing \quad \text{when } |I| = \omega,\end{aligned}$$

we have

$$f^{-1}(\{p\}) \not\subset U_I \quad \text{for } I \in \mathcal{P}(\alpha) \text{ and } |I| = \omega.$$

It follows without difficulty that $\{f^{-1}(\{p\})\backslash U_I: I \in \mathcal{P}(\alpha) \text{ and } |I| = \omega\}$ is a family of nonempty compact subsets of X with the finite intersection property. Hence there is $q \in X$ such that

$$f(q) = p, \quad \text{and} \quad q \notin U_I \quad \text{for } I \in \mathcal{P}(\alpha) \text{ and } |I| = \omega.$$

To prove that $\chi(q, X) \geqslant \alpha$ we let $\mathfrak{B}$ be a base for the neighborhoods of q and for $U \in \mathfrak{B}$ we set

$$F_U = \{\xi < \alpha: U \subset f^{-1}(V_\xi)\}.$$

Since $q \notin U_I$ whenever $|I| = \omega$ it follows that $|F_U| < \omega$ for all $U \in \mathfrak{B}$, and since $\mathfrak{B}$ is a local basis at q we have $\bigcup\{F_U: U \in \mathfrak{B}\} = \alpha$. Thus $|\mathfrak{B}| \geqslant \alpha$, as required.

2.7. THEOREM. *If $\alpha \geqslant \omega$ then $|\{p \in U(\alpha): \chi(p) = 2^\alpha\}| = 2^{2^\alpha}$.*

PROOF. Let $X = U(\alpha)$ and $Y = (\beta(\alpha))^{2^\alpha}$ and (from Lemma 2.3) let f be a continuous function from X onto Y. For $p \in Y$ and $\xi < 2^\alpha$ we set $V_\xi(p) = \pi_\xi^{-1}(\{p\})$ and we note that the family $\{V_\xi(p): \xi < 2^\alpha\}$ satisfies the hypotheses of Lemma 2.6 (with α replaced by 2^α). Thus for every $p \in Y$ there is $q \in f^{-1}(\{p\})$ such that $\chi(q, X) \geqslant 2^\alpha$. The result follows.

We note that from Theorem 2.5 and 2.7 it follows that if $\alpha \geqslant \omega$ then $\mathrm{w}(U(\alpha)) = \mathrm{w}(\beta(\alpha)\backslash\alpha) = \mathrm{w}(\beta(\alpha)) = 2^\alpha$. It is tempting to conjecture that $\chi(p) = 2^\alpha$ for every $p \in U(\alpha)$. This equality is immediate from Theorem 2.5 if the segment $2^\alpha = \alpha^+$ of the generalized continuum hypothesis is assumed, but it cannot be proved in ZFC. Indeed Kunen [**Ku2**] has defined a model of ZFC in which ($\omega^+ < 2^\omega$ and) there is $p \in U(\omega)$ such that $\chi(p) = \omega^+$.

B. THE RUDIN-KEISLER PARTIAL ORDER

The remarks in §2 suggest one crude classification of the elements of $\beta(\alpha)$: there are, first of all, the fixed ultrafilters (*i.e.*, the elements of α) and then there are, for each cardinal number γ such that $\omega \leqslant \gamma \leqslant \alpha$, the elements p of $\beta(\alpha)$ such that $\gamma = \min\{|A|: A \in p\}$. (Such ultrafilters may be called γ-*uniform* on α; in this terminology, the elements of $U(\alpha)$ are simply the α-uniform ultrafilters on α.) If $\omega \leqslant \delta < \gamma \leqslant \alpha$ then one feels that any two ultrafilters on α, one δ-uniform and the other γ-uniform, are distinguishable as families of sets (as surely they are, according to Theorem 2.5, if $2^\delta \leqslant \gamma$), but it is much less obvious whether or not every two γ-uniform ultrafilters "behave and look alike". We describe now the Rudin-Keisler (pre-) order on $\beta(\alpha)$ and we indicate how to identify ultrafilters minimal in $\beta(\alpha)\backslash\alpha$, and ultrafilters minimal in $U(\alpha)$. The existence of elements of $U(\alpha)$ which are minimal and the existence of elements of $U(\alpha)$ which are not minimal, together with the simplicity of the definition of the Rudin-Keisler order, justify the introduction of that order into the literature and prove that it is a responsible, dependable tool in the attempt to classify ultrafilters. The question asked in 6.9 concerning the "width" of $U(\alpha)$ in the Rudin-Keisler order, however, apparently simple but not yet solved, indicates that our understanding of this order is far from complete.

The basic properties of the Rudin-Keisler order were studied by M. E. Rudin [**R1**] and by Keisler [**K**]. It was apparently Katětov [**K1**], however, who first defined (for arbitrary filters) an ordering equivalent to the Rudin-Keisler order, and who proved [**K2**] Theorem 3.3 below; see also his paper [**K3**].

Closely related to the Rudin-Keisler order and the ability to distinguish (by combinatorial or topological methods) between differing ultrafilters is the question of homogeneity of the spaces $U(\alpha)$. (Recall in this connection that a space X is said to be *homogeneous* if whenever $p, q \in X$ there is a homeomorphism h of X onto X such that $h(p) = q$.) We give in §8 below a very general nonhomogeneity theorem provided by Frolík and Kunen which settles this and several related questions in the negative, but for the moment we note simply that in the model of Kunen referred to above, there are points $p, q \in U(\omega)$ such that $\chi(p) = \omega^+ < 2^\omega$ and $\chi(q) = 2^\omega$; for a strong reason, then, $U(\omega)$ in Kunen's model is not homogeneous.

3. **The relations $\sim$, $\preccurlyeq$ on $\beta(\alpha)$.** For α a cardinal we denote by α^α the set of functions from α to α, and for $f \in \alpha^\alpha$ we denote by $\bar{f}$ the *Stone extension* of $f: \alpha \to \alpha \subset \beta(\alpha)$–*i.e.*, $\bar{f}$ is that continuous function from $\beta(\alpha)$ to $\beta(\alpha)$ for which $\bar{f}|\alpha = f$. We define an equivalence relation $\sim$ on $\beta(\alpha)$ and the *Rudin-Keisler (pre-) order* $\preccurlyeq$ on $\beta(\alpha)$ as follows.

Definition. Let α be a cardinal and let $p, q \in \beta(\alpha)$. Then
(a) $p \sim q$ if there is a permutation f of α such that $\bar{f}(q) = p$; and
(b) $p \preccurlyeq q$ if there is $f \in \alpha^\alpha$ such that $\bar{f}(q) = p$.

Since a permutation of α extends to a homeomorphism of $\beta(\alpha)$ and since, conversely, the restriction to α of a homeomorphism of $\beta(\alpha)$ is a permutation of α, it is clear that $p \sim q$ if and only if there is a homeomorphism h of $\beta(\alpha)$ such that $h(q) = p$.

The relation between $\sim$ and $\preccurlyeq$ is clarified by the following result. I am grateful to Professor M. Katětov for supplying the information that an even more general result is available in the 1951 paper of de Bruijn and Erdős [**BE**]. The result was known in 1963 to Kenyon [**Kn**]; proofs are available in [**Bk**], [**CN2**, Lemma 9.1], and [**O**].

3.1. Lemma. *Let α be a cardinal and let $f \in \alpha^\alpha$ be such that $f(\xi) \neq \xi$ for $\xi < \alpha$. Then there are three disjoint subsets A_0, A_1 and A_2 of α such that*

$$\alpha = A_0 \cup A_1 \cup A_2, \quad \textit{and}$$

$$A_i \cap f[A_i] = \varnothing \quad \textit{for } 0 \leqslant i \leqslant 2.$$

3.2. Lemma. *Let $f \in \alpha^\alpha$ and $p \in \beta(\alpha)$. Then*
(a) $\bar{f}(p) = \{A \subset \alpha: f^{-1}(A) \in p\}$;
(b) $\bar{f}(p) = p$ *if and only if* $\{\xi < \alpha: f(\xi) = \xi\} \in p$; *and*
(c) $\bar{f}(p) \sim p$ *if and only if there is* $A \in p$ *such that* $f|A$ *is one-to-one.*

PROOF. (a) If $f^{-1}(A) \in p$ then $p \in \mathrm{cl}_{\beta(\alpha)} f^{-1}(A)$ and hence

$$\bar{f}(p) \in \bar{f}[\mathrm{cl}_{\beta(\alpha)} f^{-1}(A)] \subset \mathrm{cl}_{\beta(\alpha)} \bar{f}[f^{-1}(A)] \subset \mathrm{cl}_{\beta(\alpha)} A;$$

hence $A \in \bar{f}(p)$. Conversely, if $A \subset \alpha$ and $f^{-1}(A) \notin p$ then $\alpha \backslash f^{-1}(A) = f^{-1}(\alpha \backslash A) \in p$, so that $\alpha \backslash A \in \bar{f}(p)$ and $A \notin \bar{f}(p)$.

(b) Define $B = \{\xi < \alpha: f(\xi) = \xi\}$. If $B \in p$ then $p \in \mathrm{cl}_{\beta(\alpha)} B$ and, hence, $\bar{f}(p) = p$. If $B \notin p$ we choose $g \in \alpha^\alpha$ such that $g|\alpha \backslash B = f|\alpha \backslash B$ and $g|B$ is a fixed-point free function, we note that $\bar{g}(p) = \bar{f}(p)$ (because g and f agree on $\alpha \backslash B$, and $p \in \mathrm{cl}_{\beta(\alpha)} \alpha \backslash B$), and from Lemma 3.1 (applied to g) we find $A \subset \alpha$ such that $A \in p$ and $A \cap g[A] = \varnothing$. From the remark preceding Lemma 1.2 we have $\mathrm{cl}_{\beta(\alpha)} A \cap \mathrm{cl}_{\beta(\alpha)} g[A] = \varnothing$, contradicting the relations $p \in \mathrm{cl}_{\beta(\alpha)} A$, $p = \bar{f}(p) = \bar{g}(p) \in \mathrm{cl}_{\beta(\alpha)} g[A]$.

(c) If $\bar{f}(p) \sim p$ there is a permutation g of α such that $\bar{g}(p) = \bar{f}(p)$ and we have $g \circ f \in \alpha^\alpha$ and

$$(g \circ f)^{-}(p) = \bar{g}(\bar{f}(p)) = p.$$

We set $A = \{\xi < \alpha: (g \circ f)(\xi) = \xi\}$ and we note from part (b) above that $A \in p$; it is clear that $f|A$ is one-to-one. For the converse suppose that there is $A \in p$ such that $f|A$ is one-to-one, and let B be a subset of A such that $B \in p$ and $|\alpha \backslash B| = |\alpha \backslash f[B]| = \alpha$. There is a permutation g of α such that $g|B = f|B$, and since g agrees with f on $B \in p$ we have $\bar{f}(p) = \bar{g}(p) \sim p$, as required.

It is clear that the relation $\sim$ defined above on $\beta(\alpha)$ is indeed an equivalence relation. The following result makes precise the statement that the pre-order $\preccurlyeq$ respects $\sim$.

3.3. THEOREM. *Let α be a cardinal and let $p, q, r, s \in \beta(\alpha)$. Then*

(a) $p \preccurlyeq p$;

(b) *if* $p \preccurlyeq q$ *and* $q \preccurlyeq r$, *then* $p \preccurlyeq r$;

(c) *if* $p \sim q$, $q \preccurlyeq r$, *and* $r \sim s$, *then* $p \preccurlyeq s$; *and*

(d) *if* $p \preccurlyeq q$ *and* $q \preccurlyeq p$, *then* $p \sim q$.

PROOF. We need prove only (d). There are $f, g \in \alpha^\alpha$ such that $\bar{f}(p) = q$ and $\bar{g}(q) = p$, so that $(g \circ f)^{-}(p) = p$. We set $A = \{\xi < \alpha: (g \circ f)(\xi) = \xi\}$, we note that $f|A$ is one-to-one, and we note from 3.2(b) that $A \in p$; hence $q = \bar{f}(p) \sim p$ by 3.2(c).

Theorem 3.3 shows that the quotient relation defined by $\preccurlyeq$ on $\beta(\alpha)/\sim$ is indeed a partial order. We denote this relation also by $\preccurlyeq$ and somewhat carelessly we refer, except when unusual precision is demanded, to the Rudin-Keisler order $\preccurlyeq$ on $\beta(\alpha)$.

If $p, q \in \beta(\alpha)$, we write $p \prec q$ if $p \preccurlyeq q$ but $p \nsim q$.

We note finally that if $\alpha \geqslant \omega$ and $p \in \beta(\alpha)$, then $|\{q \in \beta(\alpha): q \preccurlyeq p\}| \leqslant 2^\alpha$ (and hence $|\{q \in \beta(\alpha): q \sim p\}| \leqslant 2^\alpha$). Indeed if $q \preccurlyeq p$ there is $f \in \alpha^\alpha$ such that $\bar{f}(p) = q$, and we have

$$|\{q \in \beta(\alpha): q \preccurlyeq p\}| \leqslant |\{\bar{f}(p): f \in \alpha^\alpha\}| \leqslant |\alpha^\alpha| = 2^\alpha.$$

4. Uniform ultrafilters, $\preccurlyeq$-minimal in $\beta(\alpha)\backslash\alpha$. For $\zeta < \alpha$ the function $f \in \alpha^\alpha$ defined by $f(\xi) = \zeta$ for all $\xi < \alpha$ clearly satisfies $\bar{f}(p) = \zeta$ for all $p \in \beta(\alpha)$, so that the principal ultrafilter ζ is $\preccurlyeq$-minimal in $\beta(\alpha)$; further, since $\zeta \prec p$ if $p \in \beta(\alpha)\backslash\alpha$, no nonprincipal ultrafilter is $\preccurlyeq$-minimal in $\beta(\alpha)$. We turn now to an investigation of those ultrafilters which are

(1) uniform, and $\preccurlyeq$-minimal in $\beta(\alpha)\backslash\alpha$;

(2) uniform, and $\preccurlyeq$-minimal in $U(\alpha)$.

For A a set and γ a cardinal, we write

$$[A]^\gamma = \{B \subset A: |B| = \gamma\}, \quad \text{and} \quad [A]^{<\gamma} = \{B \subset A: |B| < \gamma\}.$$

The terminology of the following definition, which differs slightly from that of **[CN2]**, is chosen in appreciation of the celebrated theorem $\omega \to (\omega)^2_2$ of Ramsey **[Ra]**; this notation means that if $[\omega]^2 = P_0 \cup P_1$, then there are $A \in [\omega]^\omega$ and $i < 2$ such that $[A]^2 \subset P_i$.

Notation. Let $\alpha \geqslant \omega$, $n < \omega$, and $p \in U(\alpha)$. Then $\alpha \to (p)^n_2$ if the following condition is satisfied: If $[\alpha]^n = P_0 \cup P_1$, then there are $A \in p$ and $i < 2$ such that $[A]^n \subset P_i$.

DEFINITION. Let $\alpha \geqslant \omega$ and $p \in U(\alpha)$. Then p is a *Ramsey* ultrafilter if $\alpha \to (p)^2_2$; and p is a *strongly Ramsey* ultrafilter if $\alpha \to (p)^n_2$ for all $n < \omega$.

For use in the proof of Theorem 4.5 we note that if $p, q \in U(\alpha)$, $p \sim q$ and q is a strongly Ramsey ultrafilter, then p is a strongly Ramsey ultrafilter. Indeed let f be a permutation of α such that $\bar{f}(q) = p$, let $[\alpha]^n = P_0 \cup P_1$ with $n < \omega$, and for $i < 2$ define $Q_i = \{f^{-1}(F): F \in P_i\}$. Since $[\alpha]^n = Q_0 \cup Q_1$ there are $A \in q$ and $i < 2$ such that $[A]^n \subset Q_i$, and then $f[A] \in p$ and $[f[A]]^n \subset P_i$, as required.

DEFINITION. Let $\alpha \geqslant \omega$ and $p \in U(\alpha)$. Then p is a *selective* ultrafilter if for every partition $\{d_\eta: \eta < \alpha\}$ of α either there is $\eta < \alpha$ such that $d_\eta \in p$ or there is $A \in p$ such that $|A \cap d_\eta| \leqslant 1$ for all $\eta < \alpha$.

It is clear that p is selective if and only if for every partition $\{d_\eta: \eta < \alpha\}$ there is $A \in p$ such that $|\{\eta < \alpha: |A \cap d_\eta| > 1\}| \leqslant 1$. The following simple result, which is preliminary to 4.5, shows that it is unusual that there exist a Ramsey ultrafilter in $U(\alpha)$; indeed, as is remarked in 4.6(a), it is consistent with ZFC that for no cardinal α is there Ramsey $p \in U(\alpha)$.

An ultrafilter is said to be γ-*complete* (with γ a cardinal) if $\bigcap \mathfrak{F} \in p$ whenever $\mathfrak{F} \subset p$ and $|\mathfrak{F}| < \gamma$. A cardinal α is *measurable* if there is a nonprincipal α-complete ultrafilter on α. According to this definition, ω is a measurable cardinal. It is consistent with ZFC that there is no uncountable measurable cardinal, and according to the Incompleteness Theorem of Gödel **[C]**, **[Cr]** it cannot be shown in ZFC that the existence of uncountable measurable cardinals is consistent with ZFC. It seems conceivable that someday someone may prove in ZFC that there is no uncountable measurable cardinal, but nevertheless most set theorists today are willing to assume the existence of large cardinals satisfying various inaccessibility properties, including the property of measurability.

4.1. THEOREM. *Let $\alpha \geqslant \omega$ and let $p \in U(\alpha)$. Then*

(a) *if p is a Ramsey ultrafilter, then p is selective;*

(b) *if p is selective, then p is $\preccurlyeq$-minimal in $\beta(\alpha)\backslash\alpha$; and*

(c) *if p is $\preccurlyeq$-minimal in $\beta(\alpha)\backslash\alpha$, then p is α-complete (and hence α is a measurable cardinal).*

PROOF. (a) Let $\{d_\eta: \eta < \alpha\}$ be a partition of α and for $\{\xi, \xi'\} \in [\alpha]^2$ define

$$\begin{aligned} \{\xi, \xi'\} &\in P_0 \quad \text{if there is } \eta < \alpha \text{ such that } \xi, \xi' \in d_\eta, \\ &\in P_1 \quad \text{otherwise.} \end{aligned}$$

There are $A \in p$ and $i < 2$ such that $[A]^2 \subset P_i$. If $[A]^2 \subset P_0$ then clearly there is $\eta < \alpha$ such that $A \subset d_\eta$, and if $[A]^2 \subset P_1$ then it is clear that $|A \cap d_\eta| \leqslant 1$ for all $\eta < \alpha$.

(b) We must show that if $f \in \alpha^\alpha$ then either $\bar{f}(p) \sim p$ or $\bar{f}(p) \in \alpha$. Let $\{d_\eta: \eta < \alpha\}$ be the partition of α defined by the rule $d_\eta = f^{-1}(\{\eta\})$. If there is $\eta < \alpha$ such that $d_\eta \in p$, then since $f(\zeta) = \eta$ for all $\zeta \in d_\eta \in p$, we have $\bar{f}(p) = \eta < \alpha$; and if there is $A \in p$ such that $|A \cap d_\eta| \leqslant 1$ for all $\eta < \alpha$, then clearly $f|A$ is a one-to-one function and hence $\bar{f}(p) \sim p$ by 3.2(c).

(c) Suppose that there is $\{A_\xi: \xi < \gamma\} \subset p$ with $\gamma < \alpha$ such that $\bigcap_{\xi<\gamma} A_\xi \notin p$. We may assume without loss of generality, replacing each of the sets A_ξ by $A_\xi \backslash \bigcap_{\zeta<\gamma} A_\zeta$, that $\bigcap_{\xi<\gamma} A_\xi = \varnothing$. Now for $\zeta < \alpha$ let $f(\zeta)$ be chosen so that $\zeta \notin A_{f(\zeta)}$. Since p is $\preccurlyeq$-minimal in $\beta(\alpha)\backslash\alpha$ and $f \in \gamma^\alpha \subset \alpha^\alpha$, we have either $\bar{f}(p) \in \alpha$ or $\bar{f}(p) \sim p$. If there is $\eta < \alpha$ such that $\bar{f}(p) = \eta$, then from 3.2(a) we have

$$\varnothing = A_\eta \cap f^{-1}(\{\eta\}) \in p,$$

a contradiction; and if $\bar{f}(p) \sim p$ then from 3.2(c) there is $A \in p$ such that $f|A$ is one-to-one, contradicting the facts that $|A| = \alpha$ (since $p \in U(\alpha)$) and $f[A] \subset \gamma < \alpha$.

To continue the study of uniform ultrafilters on α which are $\preccurlyeq$-minimal in $\beta(\alpha)\backslash\alpha$ it is convenient to consider separately the cases $\alpha > \omega$, $\alpha = \omega$. In either case, the following definition and lemma, due to Rowbottom **[Ro]**, are helpful.

DEFINITION. Let $\alpha \geqslant \omega$ and $p \in U(\alpha)$. Then p is *quasi-normal* if for every family $\{A_\xi: \xi < \alpha\} \subset p$ there is $A \in p$ such that if $\xi, \zeta \in A$ and $\xi < \zeta$ then $\zeta \in A_\xi$.

4.2. LEMMA. *Let $\alpha \geqslant \omega$ and let p be a quasi-normal ultrafilter on α. Then p is a strongly Ramsey ultrafilter.*

PROOF. It is clear that $\alpha \to (p)^0_2$ and $\alpha \to (p)^1_2$. We assume that $n < \omega$ and $\alpha \to (p)^n_2$, and we prove that $\alpha \to (p)^{n+1}_2$.

Let $[\alpha]^{n+1} = P_0 \cup P_1$, and for $\xi < \alpha$ define $P_0(\xi) = \{F \in [\alpha]^n$: either $\xi \in F$, or $\xi \notin F$ and $F \cup \{\xi\} \in P_0\}$, and $P_1(\xi) = [\alpha]^n \backslash P_0(\xi)$. By the assumption $\alpha \to (p)_2^n$ there are $A_\xi \in p$ and $\varepsilon(\xi) < 2$ such that $[A_\xi]^n \subset P_{\varepsilon(\xi)}(\xi)$, and since p is quasi-normal there is $A \in p$ such that if $\xi, \zeta \in A$ with $\xi < \zeta$ then $\zeta \in A_\xi$. We assume without loss of generality, by considering if necessary the sets $\{\xi \in A: \varepsilon(\xi) = 0\}$ and $\{\xi \in A: \varepsilon(\xi) = 1\}$ (of which one belongs to p), that there is $i < 2$ such that $\varepsilon(\xi) = i$ for all $\xi \in A$.

We claim that $[A]^{n+1} \subset P_i$. Indeed let $G \in [A]^{n+1}$ and define $\xi = \min G$ and $F = G \backslash \{\xi\}$. Then $F \in [A]^n$, and if $\zeta \in F$ then $\xi < \zeta$ (and hence $\zeta \in A_\xi$); thus $F \in [A_\xi]^n \subset P_i(\xi)$, so that $G = F \cup \{\xi\} \in P_i$, as required.

The following two lemmas allow us in Theorem 4.5 to complete the cycle characterizing the uniform ultrafilters on α which are $\preccurlyeq$-minimal in $\beta(\alpha)\backslash\alpha$. The ultraproduct argument used in the first of these (Lemma 4.3) is due to Scott [KT]. The argument in Lemma 4.4, due to Kunen, is taken from Blass [Bl, Proposition 10.6]. As Blass shows (see also the implication (d) $\Rightarrow$ (i) of Theorem 9.6 of [CN2]) the proof of Lemma 4.4 adapts with only minor modifications to the case of an arbitrary (possibly uncountable) measurable cardinal. We note that this stronger version of Lemma 4.4, though easily achieved, would not allow us to omit Lemma 4.3 from the present discussion: It is 4.3 that gives the existence of $\preccurlyeq$-minimal ultrafilters in $\beta(\alpha)\backslash\alpha$ for uncountable, measurable α.

4.3. LEMMA. *Let $\alpha > \omega$ and let p be an α-complete nonprincipal ultrafilter on α. Then there is $f \in \alpha^\alpha$ such that $\bar{f}(p)$ is quasi-normal.*

PROOF. For $f, g \in \alpha^\alpha$ we write $f =_p g$ if $\{\xi < \alpha: f(\xi) = g(\xi)\} \in p$; clearly $=_p$ is an equivalence relation on α^α. For $f \in \alpha^\alpha$ we denote by f/p the $=_p$ equivalence class of f, and we set

$$\alpha^\alpha/p = \{f/p: f \in \alpha^\alpha\}.$$

Next for $f, g \in \alpha^\alpha$ we write

$$f/p \leqslant_p g/p \quad \text{if } \{\xi < \alpha: f(\xi) \leqslant g(\xi)\} \in p;$$

clearly the relation $\leqslant_p$ is well defined on α^α/p (in the sense that if $f' =_p f$, $g' =_p g$ and $f/p \leqslant_p g/p$, then $f'/p \leqslant g'/p$), and $\leqslant_p$ is a linear order on α^α/p, since if $f, g \in \alpha^\alpha$, then one of the three sets

$$\{\xi < \alpha: f(\xi) < g(\xi)\}, \quad \{\xi < \alpha: f(\xi) = g(\xi)\}, \quad \{\xi < \alpha: f(\xi) > g(\xi)\}$$

is an element of p. We note that in fact α^α/p is well ordered by $\leqslant_p$. Otherwise there are $\{f_n: n < \omega\} \subset \alpha^\alpha$ and $\{A_n: n < \omega\} \subset p$ such that

$$\text{if } \xi \in A_n \text{ then } f_{n+1}(\xi) < f_n(\xi);$$

then $\bigcap_{n<\omega} A_n \in p$ (since p is ω^+-complete) and for $\xi \in \bigcap_{n<\omega} A_n$ the ordinal sequence $\{f_n(\xi): n < \omega\}$ descends strictly, a contradiction.

Since α^α/p is well ordered by $\leqslant_p$, there is a (unique) ordinal number $\bar{\xi}(p)$ and a (unique) order-isomorphism φ from α^α/p onto $\bar{\xi}(p)$.

Let e be the natural embedding of α into α^α; that is, for $\zeta < \alpha$ define the function $e(\zeta) \in \alpha^\alpha$ by $e(\zeta)(\eta) = \zeta$ (for $\eta < \alpha$). It is easy to check that the composition $\varphi \circ e$ is the natural inclusion of α into $\bar{\xi}(p)$. We claim further (denoting by id the identity function of α) that $\varphi(\mathrm{id}/p) \geqslant \alpha$. Indeed if there is $\zeta < \alpha$ such that

$$\varphi(\mathrm{id}/p) = \zeta = \varphi(e(\xi)/p),$$

then

$$\{\zeta\} = \{\xi < \alpha\colon \mathrm{id}(\xi) = e(\zeta)(\xi)\} \in p$$

and p is the principal ultrafilter $\zeta < \alpha$, a contradiction.

Now we choose and fix $f \in \alpha^\alpha$ so that $\varphi(f/p) = \alpha$; that is, f satisfies

$$(*) \qquad e(\zeta)/p <_p f/p \quad \text{for all } \zeta < \alpha,$$

and f/p is minimal in α^α/p with respect to $(*)$.

To show that the ultrafilter $\bar{f}(p)$ is quasi-normal, we first verify that $\bar{f}(p) \in U(\alpha)$. If there is $A \in \bar{f}(p)$ such that $|A| < \alpha$, then since $f^{-1}(A) \in p$ and p is α-complete there is $\zeta \in A$ such that $f^{-1}(\{\zeta\}) \in p$; but then $e(\zeta) =_p f$, a contradiction.

Finally let $\{A_\xi\colon \xi < \alpha\} \subset \bar{f}(p)$, define

$$A = \{\zeta < \alpha\colon \zeta \in A_\xi \text{ for all } \xi < \zeta\},$$

and define $g \in \alpha^\alpha$ as follows:

$$\begin{aligned} g(\zeta) &= f(\zeta) && \text{if } f(\zeta) \in A, \\ &= \min\{\xi\colon f(\zeta) \notin A_\xi\} && \text{if } f(\zeta) \notin A. \end{aligned}$$

We claim that $f^{-1}(A) \in p$. If the claim fails then

$$\{\zeta < \alpha\colon g(\zeta) < f(\zeta)\} \supset \alpha \backslash f^{-1}(A) \in p,$$

and from the minimality of f/p with respect to $(*)$ there are $\eta < \alpha$ and $B \in p$ such that $(B \subset \alpha\backslash f^{-1}(A)$ and) for all $\zeta \in B$ we have $g(\zeta) = \eta$ (and hence $f(\zeta) \notin A_\eta$); since $B \in p$ we have $f[B] \in \bar{f}(p)$ and hence

$$\varnothing = A_\eta \cap f[B] \in \bar{f}(p),$$

a contradiction. It follows that $f^{-1}(A) \in p$, so that $A \in \bar{f}(p)$. It is clear that if $\xi, \zeta \in A$ and $\xi < \zeta$, then $\zeta \in A_\xi$.

4.4. Lemma. *If p is a $\preccurlyeq$-minimal element of $U(\omega)$, then p is quasi-normal.*

Proof. We show that if $\{A_k\colon k < \omega\} \subset p$ then there is $A \in p$ such that if $m, k \in A$ and $m < k$ then $k \in A_m$. We assume without loss of generality, replacing if necessary A_k by $A_k \backslash \{k\}$, that $\bigcap_{k<\omega} A_k = \varnothing$, and for $k < \omega$ we define

$$f(k) = \min\{m < \omega : k \notin A_m\}.$$

Then $f \in \omega^\omega$ and, hence, $\bar{f}(p) \in \omega$ or $\bar{f}(p) \sim p$. If $\bar{f}(p) = m < \omega$ then from 3.2(a) we have $f^{-1}(\{m\}) \in p$ and hence

$$\varnothing = f^{-1}(\{m\}) \cap A_m \in p,$$

a contradiction. Hence $\bar{f}(p) \sim p$ and from 3.2(c) there is $D \in p$ such that $f|D$ is one-to-one.

For $m < \omega$ we have $|\{k \in D : f(k) \leqslant m\}| < \omega$; we set

$$g(m) = \max\{m + 1, \max\{k \in D : f(k) \leqslant m\}\},$$

we note that $g(m) \geqslant m + 1$ and that $g(m) \leqslant g(m')$ if $m < m' < \omega$, and we define $h \in \omega^\omega$ by the rule

$$h(0) = 0, \quad h(m + 1) = g(h(m)) \quad \text{for } 0 < m < \omega.$$

Since $h(m + 1) \geqslant h(m) + 1$, we have $\sup\{h(m) : m < \omega\} = \omega$; hence we may define $e \in \omega^\omega$ by the rule

$$e(m) = \min\{k : m \leqslant h(k)\}.$$

Concerning the function e we note three facts: $e(m) \leqslant e(m')$ if $m < m' < \omega$; $m \leqslant h(e(m))$ for all $m < \omega$; and if $e(m) = k$ then $m \leqslant h(k)$ (and hence

$$|e^{-1}(\{k\})| \leqslant h(k) + 1 < \omega \text{ for all } k < \omega).$$

Thus $\bar{e}(p) \notin \omega$, so $\bar{e}(p) \sim p$ and there is $C \in p$ such that $e|C$ is one-to-one. We set $B = C \cap D$ and we enumerate B in its natural (increasing) order as follows:

$$B = \{n_i : i < \omega\}, \qquad n_i < n_{i+1} \text{ for } i < \omega.$$

We define

$$B_0 = \{n_i : i \text{ is even}\} \quad \text{and} \quad B_1 = \{n_i : i \text{ is odd}\},$$

and we denote by A whichever of the two sets B_0, B_1 is an element of p. To see that A is as required let $m, k \in A$ with $m < k$, choose $n \in B \backslash A$ such that $m < n < k$ and note that since e preserves order and is one-to-one on B we have $e(m) + 1 < e(k)$ and hence $h(e(m) + 1) < k$; thus $g(h(e(m))) < k$, so that $m \leqslant h(e(m)) < f(k)$ and hence $k \in A_m$, as required.

The following theorem summarizes the (positive) results of this section.

4.5. THEOREM. *Let $\alpha \geqslant \omega$ and $p \in U(\alpha)$. The following conditions are equivalent, and each implies that α is a measurable cardinal*:

(a) *p is a strongly Ramsey ultrafilter*;

(b) *p is a Ramsey ultrafilter*;

(c) *p is a selective ultrafilter*; *and*

(d) *p is $\leqslant$-minimal in $\beta(\alpha)\backslash\alpha$.*

Furthermore: *If α is a measurable cardinal and $\alpha > \omega$, then there is $p \in U(\alpha)$ such that p satisfies these conditions.*

PROOF. (a) $\Rightarrow$ (b) is obvious; (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d), together with the statement that (d) implies that α is a measurable cardinal, are given by 4.1.

To prove that (d) $\Rightarrow$ (a) we consider separately the cases $\alpha = \omega$, $\alpha > \omega$. If $\alpha = \omega$ then p is quasi-normal by 4.4 and from 4.2 we have (a). If $\alpha > \omega$ then from 4.1(c) and 4.3 there is $f \in \alpha^\alpha$ such that $\bar{f}(p)$ is quasi-normal; then $\bar{f}(p)$ is a strongly Ramsey ultrafilter by 4.2. From the fact that p is $\leqslant$-minimal in $\beta(\alpha)\backslash\alpha$ (and $\bar{f}(p) \in U(\alpha) \subset \beta(\alpha)\backslash\alpha$) we have $p \sim \bar{f}(p)$. Hence p is $\sim$-equivalent to a strongly Ramsey ultrafilter and is therefore itself a strongly Ramsey ultrafilter.

The final statement of the theorem is proved (indeed, improved) by this, which follows from 4.3 and 4.2: If $\alpha > \omega$ and (α is measurable and) q is an α-complete uniform ultrafilter on α, then there is $p \in U(\alpha)$ such that $p \leqslant q$ and p satisfies conditions (a) through (d).

4.6. REMARKS. (a) For the (measurable) cardinal ω, Theorem 4.5 may be viewed as offering some characterizations of the $\leqslant$-minimal uniform ultrafilters but it should be noted that it fails to assert their existence. There is good reason for this: Kunen ([**Ku2**, remark following Theorem 2.2] and [**Ku3**, §5]) has shown that there is a model of ZFC in which there is no $\leqslant$-minimal element of $U(\omega)$. The usual device of excluding from a model of ZFC all sets whose rank is at least as great as the least uncountable, measurable cardinal may be applied in particular to Kunen's model, and we have the following statement: It is equiconsistent with ZFC that for every infinite cardinal α no element of $U(\alpha)$ is $\leqslant$-minimal in $\beta(\alpha)\backslash\alpha$.

We note in 5.5 below, however, that it is also consistent with ZFC that elements of $U(\alpha)$ which are $\leqslant$-minimal in $U(\alpha)$ exist in profusion.

(b) The proof given of Theorem 4.5 shows that the four conditions stated are equivalent also to the condition that the ultrafilter p be $\sim$-equivalent to a quasi-normal ultrafilter. In fact an ultrafilter $\sim$-equivalent to a quasi-normal ultrafilter is itself quasi-normal, so this latter condition may be added to the list of equivalent conditions in Theorem 4.5. A proof of this, together with several other additional equivalent conditions, is given in Theorem 9.6 of [**CN2**].

(c) For the proofs of portions of Theorem 4.5 due to Kunen, see the doctoral dissertations of Blass [**Bl**] and Booth [**Bo**]. See also Frolík [**F5**, Theorem 4.4.5] for the equivalence 4.5(c) $\Leftrightarrow$ (d).

5. **Uniform ultrafilters $\leqslant$-minimal in** $U(\alpha)$. In this section as in §4, we present partial results less definitive than the full statements offered in [**CN2**]. Again it is hoped that the relatively unencumbered arguments given here will be appealing, and will simplify, for those interested in the fuller account, the reading of [**CN2**].

We have noted in §4 that for $\alpha \geqslant \omega$ a uniform ultrafilter p on α is selective if for every partition $\{d_\eta : \eta < \alpha\}$ of α there is $A \in p$ such that

$$|\{\eta < \alpha : |A \cap d_\eta| > 1\}| \leqslant 1.$$

We now define a weaker concept.

Let $\alpha \geqslant \omega$ and $p \in U(\alpha)$. Then p is *uniformly selective* if for every partition $\{d_\eta : \eta < \alpha\}$ of α there is $A \in p$ such that

$$|\{\eta < \alpha : |A \cap d_\eta| > 1\}| < \alpha.$$

5.1. LEMMA. *Let $\alpha \geqslant \omega$ and $p \in U(\alpha)$. Then the following statements are equivalent.*

(a) *p is $\preccurlyeq$-minimal in $U(\alpha)$;*

(b) *p is uniformly selective.*

PROOF. (a) $\Rightarrow$ (b). Let $\{d_\eta : \eta < \alpha\}$ be a partition of α, and for $\xi < \alpha$ let $f(\xi) = \eta$ if $\xi \in d_\eta$. Then $f \in \alpha^\alpha$, and from condition (a) it follows that either $\bar{f}(p) \notin U(\alpha)$ or $\bar{f}(p) \sim p$. In the former case there is $B \in \bar{f}(p)$ such that $|B| < \alpha$, and defining $A = f^{-1}(B)$ we have $A \in p$ and

$$|\{\eta < \alpha : |A \cap d_\eta| > 1\}| \leqslant |B| < \alpha.$$

In the latter case by 3.2(c) there is $A \in p$ such that $f|A$ is one-to-one and we have $|A \cap d_\eta| \leqslant 1$ for all $\eta < \alpha$.

(b) $\Rightarrow$ (a). We show that if $f \in \alpha^\alpha$ and $\bar{f}(p) \in U(\alpha)$, then $\bar{f}(p) \sim p$. Set $d_\eta = f^{-1}(\{\eta\})$ for $\eta < \alpha$, choose $A \in p$ such that $|\{\eta < \alpha : |A \cap d_\eta| > 1\}| < \alpha$, and define $B = \{\eta < \alpha : |A \cap d_\eta| > 1\}$. Then $B \notin \bar{f}(p)$ (since $\bar{f}(p) \in U(\alpha)$), so

$$\alpha \backslash B = \{\eta < \alpha : |A \cap d_\eta| \leqslant 1\} \in \bar{f}(p);$$

hence $A \cap f^{-1}(\alpha \backslash B) \in p$, and f is one-to-one on this set. It follows from 3.2(c) that $\bar{f}(p) \sim p$.

Notation. Let $\alpha \geqslant \omega$ and $A \in \mathcal{P}(\alpha)$. Then

$$\hat{A} = U(\alpha) \cap \mathrm{cl}_{\beta(\alpha)} A.$$

It is clear that if $p \in U(\alpha)$ then $\{\hat{A} : A \in p\}$ is a base for p in $U(\alpha)$.

For $\alpha \geqslant \omega$, a family $\mathcal{Q}$ of subsets of α is said to have the *α-uniform finite intersection property* (briefly: the uniform fip) if $|\bigcap \mathcal{F}| = \alpha$ whenever $\mathcal{F} \subset \mathcal{Q}$ and $|\mathcal{F}| < \omega$.

5.2. LEMMA. *If $\alpha \geqslant \omega$ and $\mathcal{Q}$ is a family of subsets of α with the uniform* fip *such that $|\mathcal{Q}| \leqslant \alpha$, then there is $A \in \mathcal{P}(\alpha)$ such that both $\mathcal{Q} \cup \{A\}$ and $\mathcal{Q} \cup \{\alpha \backslash A\}$ have the uniform* fip.

PROOF. If the result fails there is a unique $p \in U(\alpha)$ such that $\mathcal{Q} \subset p$, and then clearly $\{\hat{A} : A \in \mathcal{Q}\}$ is a base for p in $U(\alpha)$; this contradicts Theorem 2.5.

5.3. LEMMA. *Let $\alpha \geqslant \omega$ and let $\{A_\xi: \xi < \alpha\}$ be a family of sets such that $|A_\xi| = \alpha$ for $\xi < \alpha$. Then there is a family $\{B_\xi: \xi < \alpha\}$ of pairwise disjoint sets such that $B_\xi \subset A_\xi$ and $|B_\xi| = \alpha$ for $\xi < \alpha$.*

PROOF. We proceed by transfinite recursion. Let $p_{0,0} \in A_0$. Let $\zeta < \alpha$ and suppose that for $\xi' \leqslant \xi < \zeta$ we have defined $p_{\xi',\xi} \in A_{\xi'}$. We choose $p_{\zeta',\zeta}$ for $\zeta' \leqslant \zeta$ such that

$$p_{\zeta',\zeta} \in A_{\zeta'} \backslash \{p_{\xi',\xi}: \xi' \leqslant \xi < \zeta\} \quad \text{and}$$

$$p_{\zeta',\zeta} \neq p_{\zeta'',\zeta} \quad \text{for } \zeta' < \zeta'' \leqslant \zeta,$$

and we set

$$B_\xi = \{p_{\xi,\zeta}: \xi \leqslant \zeta < \alpha\} \quad \text{for } \xi < \alpha.$$

It is easily verified that the family $\{B_\xi: \xi < \alpha\}$ is as required.

5.4. LEMMA. *Let $\alpha \geqslant \omega$ and let $\mathscr{A}$ be a family of subsets of α with the uniform* fip *such that $|\mathscr{A}| \leqslant \alpha$. If $\{d_\eta: \eta < \alpha\}$ is a partition of α then there are disjoint subsets $A(0)$, $A(1)$ of α such that*

(a) $\mathscr{A} \cup \{A(i)\}$ *has the uniform* fip $(i < 2)$; *and*

(b) $|\{\eta < \alpha: |A(i) \cap d_\eta| > 1\}| < \alpha \quad (i < 2)$.

PROOF. *Case* 1. There is $F \in \mathscr{A}$ such that $|\{\eta < \alpha: |F \cap d_\eta| > 1\}| < \alpha$. In this case we find $A \in \mathscr{P}(\alpha)$ as in the conclusion of Lemma 5.2 and we set

$$A(0) = F \cap A, \quad \text{and} \quad A(1) = F \cap (\alpha \backslash A).$$

Case 2. $|\{\eta < \alpha: |F \cap d_\eta| > 1\}| = \alpha$ for all $F \in \mathscr{A}$. We assume without loss of generality that $\mathscr{A}$ is closed under finite intersections, we write $\mathscr{A} = \{A_\xi: \xi < \alpha\}$, and for $\xi < \alpha$ we define

$$B_\xi = \{\eta < \alpha: |A_\xi \cap d_\eta| > 1\};$$

since $|B_\xi| = \alpha$ for $\xi < \alpha$ there is by Lemma 5.3 a family $\{C_\xi: \xi < \alpha\}$ of pairwise disjoint subsets of α such that $C_\xi \subset B_\xi$ and $|C_\xi| = \alpha$ for $\xi < \alpha$.

For $\xi < \alpha$ and $\eta \in B_\xi$ we choose distinct elements $p_{\xi,\eta}(0), p_{\xi,\eta}(1)$ of $C_\xi \cap d_\eta$, and we set

$$A(i) = \{p_{\xi,\eta}(i): \xi < \alpha, \eta \in B_\xi\} \qquad (i < 2).$$

It is clear that $A(0) \cap A(1) = \varnothing$. If η, η' are distinct elements of B_ξ then $p_{\xi,\eta}(i), p_{\xi,\eta'}(i)$ are distinct elements of $A_\xi \cap A(i)$, and hence

$$|A_\xi \cap A(i)| \geqslant |B_\xi| = \alpha;$$

since $\mathscr{A}$ is closed under finite intersections, it follows that $\mathscr{A} \cup \{A(i)\}$ has the uniform fip. It remains, finally, to show that $|\{\eta < \alpha: |A(i) \cap d_\eta| > 1\}| < \alpha$. Since $p_{\xi,\eta}(i) \in A(i) \cap d_\eta$ implies $\eta \in C_\xi$, we have in fact

$$|A(i) \cap d_\eta| \leqslant |\{\xi < \alpha: \eta \in C_\xi\}| \leqslant 1 \quad \text{for all } \eta < \alpha.$$

5.5. THEOREM. *Let $\alpha \geqslant \omega$ and assume $\alpha^+ = 2^\alpha$. Then there is $S \subset U(\alpha)$ such that $|S| = 2^{2^\alpha}$ and the elements of S are $\preccurlyeq$-minimal in $U(\alpha)$.*

PROOF. Let $\mathscr{Q}$ be a family of subsets of α with the uniform fip such that $|\mathscr{Q}| \leqslant \alpha$ (for example, let $\mathscr{Q} = \{\alpha\}$), and let $\{d(\xi): \xi < 2^\alpha\}$ be a well-ordering of the set of partitions of α, with $d(\xi) = \{d(\xi)_\eta: \eta < \alpha\}$.

For $f \in 2^{(\alpha^+)}$ we define as follows a family $\{\mathscr{Q}_\xi(f): \xi < \alpha^+\}$ of subsets of α.

$\mathscr{Q}_0(f) = \mathscr{Q}$;

if $\xi < \alpha^+$ and ξ is a limit ordinal and $\mathscr{Q}_\zeta(f)$ has been defined for all $\zeta < \xi$, then $\mathscr{Q}_\xi(f) = \bigcup_{\zeta<\xi}\mathscr{Q}_\zeta(f)$; and

if $\xi < \alpha^+$ and $\mathscr{Q}_\xi(f)$ has been defined, and if $A_\xi(0), A_\xi(1)$ are disjoint subsets of α such that

$\mathscr{Q}_\xi(f) \cup \{A_\xi(i)\}$ has the uniform fip ($i < 2$), and

$|\{\eta < \alpha: |A_\xi(i) \cap d(\xi)_\eta| > 1\}| < \alpha$ ($i < 2$),

then $\mathscr{Q}_{\xi+1}(f) = \mathscr{Q}_\xi(f) \cup \{A_\xi(f(\xi))\}$.

It is easy to verify by induction that $|\mathscr{Q}_\xi(f)| \leqslant \alpha$ for all $\xi < \alpha^+$, and that each of the families $\mathscr{Q}_\xi(f)$ has the uniform fip; the sets $A_\xi(i)$ are defined using Lemma 5.2.

We note that $\mathscr{Q}_\xi(f) \subset \mathscr{Q}_{\xi'}(f)$ whenever $\xi < \xi' < \alpha^+$. For $f \in 2^{(\alpha^+)}$ we choose $p_f \in U(\alpha)$ such that $\bigcup_{\xi<\alpha^+}\mathscr{Q}_\xi(f) \subset p_f$, and we set $S = \{p_f: f \in 2^{(\alpha^+)}\}$.

To verify that p_f is a $\preccurlyeq$-minimal element of $U(\alpha)$ we note that if $d(\xi) = \{d(\xi)_\eta: \eta < \alpha\}$ is a partition of α then there is $A_\xi(f(\xi)) \in \mathscr{Q}_{\xi+1}(f) \subset p_f$ such that $|\{\eta < \alpha: |A_\xi(f(\xi)) \cap d(\xi)_\eta| > 1\}| < \alpha$.

To verify that $|S| = 2^{2^\alpha}$ it is enough to show that if f and g are distinct elements of $2^{(\alpha^+)}$ then $p_f \neq p_g$. There is $\xi < \alpha^+$ such that $f(\xi) \neq g(\xi)$, and the desired conclusion follows from the relations $A_\xi(f(\xi)) \cap A_\xi(g(\xi)) = \varnothing$, $A_\xi(f(\xi)) \in p_f$, and $A_\xi(g(\xi)) \in p_g$.

6. **The $\preccurlyeq$-width and height of $\beta(\alpha)$.** We begin with a consequence of Theorem 5.5.

6.1. THEOREM. *Let $\alpha \geqslant \omega$ and assume $\alpha^+ = 2^\alpha$.*

(a) *There is $T \subset U(\alpha)$ such that $|T| = 2^{2^\alpha}$ and the elements of T are $\preccurlyeq$-minimal in $U(\alpha)$ and pairwise $\preccurlyeq$-incomparable.*

(b) *For every $p \in U(\alpha)$ there is $T(p) \subset U(\alpha)$ such that $p \in T(p)$, $|T(p)| = 2^{2^\alpha}$, and the elements of $T(p)$ are pairwise $\preccurlyeq$-incomparable.*

PROOF. (a) From Theorem 5.5 there is $S \subset U(\alpha)$ such that $|S| = 2^{2^\alpha}$ and the elements of S are $\preccurlyeq$-minimal in $U(\alpha)$. Let T be a maximal set of pairwise incomparable elements of S. Since $|\{q \in S: q \sim p\}| \leqslant 2^\alpha$ for each $p \in S$, we have $|T| = 2^{2^\alpha}$, as required.

(b) Let T be defined as in part (a) and let $p \in U(\alpha)$. Since $|\{q \in T: q \preccurlyeq p\}| \leqslant 2^\alpha$, the set $T(p)$ defined by

$$T(p) = (T \backslash \{q \in T: q \leqslant p\}) \cup \{p\}$$

is as required.

I do not know if the conclusions of Theorem 6.1 concerning pairwise $\leqslant$-incomparability can be established in ZFC (without special set-theoretic assumptions). An alternative adequate hypothesis, incompatible with the assumption $\alpha^+ = 2^\alpha$, is given in 6.3 below. We need the following remarkable result of Hajnal [**Hj**]; his proof is given also in [**CN2**, Theorem 10.14].

6.2. Lemma. *Let $\omega \leqslant \kappa < \alpha$ and let f be a function from α to $[\alpha]^{<\kappa}$ such that if $\xi < \alpha$ then $\xi \notin f(\xi)$. Then there is $A \in [\alpha]^\alpha$ such that if $\xi, \zeta \in A$ then $\xi \notin f(\zeta)$.*

6.3. Theorem. *Let $\alpha \geqslant \omega$ and assume $(2^\alpha)^+ < 2^{2^\alpha}$. If $S \subset \beta(\alpha)$ and $|S| = 2^{2^\alpha}$, then there is $T \subset S$ such that $|T| = 2^{2^\alpha}$ and the elements of T are pairwise $\leqslant$-incomparable.*

Proof. We recall that if $p \in \beta(\alpha)$ then $|\{q \in \beta(\alpha): q \sim p\}| \leqslant 2^\alpha$. Hence we may assume without loss of generality that the elements of S are pairwise $\sim$-inequivalent. Now for $p \in S$ we set $f(p) = \{q \in S: q \prec p\}$, and we note that for $p \in S$ we have

$$|f(p)| \leqslant 2^\alpha < (2^\alpha)^+ < 2^{2^\alpha}, \quad \text{and} \quad p \notin f(p).$$

From Lemma 6.2, with κ and α replaced by $(2^\alpha)^+$ and 2^{2^α} respectively, there is $T \subset S$ such that $|T| = 2^{2^\alpha}$ and if $p, q \in T$ then $p \notin f(q)$ and $q \notin f(p)$.

The following result, concerning a "directedness" property of $\beta(\alpha)$ relative to the order $\leqslant$, is one of several given in [**CN1**]; see also [**CN2**, especially §§10.9–10.13]. The required argument has been found independently by Katětov [**K3**, Proposition 1.11].

6.4. Theorem. *Let $\alpha \geqslant \omega$ and let $S \subset \beta(\alpha)$ be such that $|S| \leqslant 2^\alpha$. Then there is $q \in U(\alpha)$ such that $S \prec q$ (i.e., such that $p \prec q$ for each $p \in S$).*

Proof. Repeating elements if necessary, we write $S = \{p_\eta: \eta < 2^\alpha\}$; this defines a point

$$p = \langle p_\eta: \eta < 2^\alpha \rangle \in (\beta(\alpha))^{2^\alpha}.$$

According to Lemma 2.3 there is $f: \alpha \to \alpha^{(2^\alpha)}$ for which the Stone extension, $\bar{f}: \beta(\alpha) \to (\beta(\alpha))^{2^\alpha}$, satisfies $\bar{f}[U(\alpha)] = (\beta(\alpha))^{2^\alpha}$. We choose $r \in U(\alpha)$ such that $\bar{f}(r) = p$, for $\eta < 2^\alpha$ we denote by π_η the projection from $\alpha^{(2^\alpha)}$ onto the ηth coordinate space α, and we note that for $\eta < 2^\alpha$ we have $\pi_\eta \circ f \in \alpha^\alpha$ and

$$(\pi_\eta \circ f)^{-}(r) = \bar{\pi}_\eta(\bar{f}(r)) = \bar{\pi}_\eta(p) = p_\eta.$$

It follows that $S \leqslant r$. We recall that $|\{t \in \beta(\alpha): t \leqslant r\}| \leqslant 2^\alpha$, and hence there is $t \in \beta(\alpha)$ such that $t \not\leqslant r$. From the argument above there is $q \in U(\alpha)$ such that $r \leqslant q$ and $t \leqslant q$. It is clear that $S \prec q$, as required.

As usual, we say that a subset S of $\beta(\alpha)$ is $\leqslant$-*cofinal* (in $\beta(\alpha)$) if for every $p \in \beta(\alpha)$ there is $q \in S$ such that $p \leqslant q$.

6.5. THEOREM. *Let $\alpha \geqslant \omega$. The following statements are equivalent.*

(a) $(2^\alpha)^+ = 2^{2^\alpha}$;

(b) *there is a $\preccurlyeq$-cofinal, linearly ordered subset S of $\beta(\alpha)$ such that* $|S| = 2^{2^\alpha}$;

(c) *there is a $\preccurlyeq$-cofinal, linearly ordered subset of $\beta(\alpha)$;*

(d) *there is $S \subset U(\alpha)$ such that $|S| = 2^{2^\alpha}$ and S has no pairwise $\preccurlyeq$-incomparable subset of cardinality 2^{2^α}.*

PROOF. (a) $\Rightarrow$ (b). Let $\beta(\alpha) = \{p_\zeta : \zeta < (2^\alpha)^+\}$, choose $q_0 \in U(\alpha)$ such that $p_0 \prec q_0$ and recursively for $\xi < (2^\alpha)^+$, q_ζ having been chosen for all $\zeta < \xi$, choose $q_\xi \in U(\alpha)$ such that

$$(\{p_\zeta : \zeta \leqslant \xi\} \cup \{q_\zeta : \zeta < \xi\}) \prec q_\xi.$$

Then $\{q_\xi : \xi < (2^\alpha)^+\}$ is a $\preccurlyeq$-cofinal, linearly ordered (indeed, well-ordered) subset of $U(\alpha)$ of cardinality $(2^\alpha)^+ = 2^{2^\alpha}$.

(b) $\Rightarrow$ (c). This is clear.

(c) $\Rightarrow$ (b). Every $\preccurlyeq$-cofinal subset S of $\beta(\alpha)$, whether or not linearly ordered, satisfies $|S| = 2^{2^\alpha}$. Indeed, writing

$$P(q) = \{p \in \beta(\alpha) : p \preccurlyeq q\} \quad \text{for } q \in S,$$

we have $\beta(\alpha) = \bigcup\{P(q) : q \in S\}$ and $|P(q)| \leqslant 2^\alpha$ for all $q \in S$; hence $|S| = 2^{2^\alpha}$.

(b) $\Rightarrow$ (d). This is clear. Indeed the linearly ordered set S of (b) has no $\preccurlyeq$-incomparable subset of cardinality 2.

(d) $\Rightarrow$ (a). This is given by Theorem 6.3.

It is apparently unknown (see 6.9 below) whether it can be shown in ZFC, without special set-theoretic assumptions, that there is $S \subset U(\alpha)$ such that $|S| = 2^{2^\alpha}$ and the elements of S are pairwise $\preccurlyeq$-incomparable. Even the innocent statement that there are two incomparable elements of $\beta(\alpha)$ (*i.e.*, that $\preccurlyeq$ is not a linear order on $\beta(\alpha)$) is unexpectedly difficult to prove; the argument below, using the concept here called $\mathcal{G}$-independence of a family of subsets of α (where $\mathcal{G}$ is a filter on α), is due to Kunen [**Ku2**]. It is desirable to find a more direct proof of Theorem 6.8, or at least of the statement that $\preccurlyeq$ is not a linear order on $\beta(\alpha)$.

Although above we have without hesitation discussed ultrafilters on α, it seems necessary now (in order to avoid confusion) to define a filter.

DEFINITION. A family $\mathcal{F}$ of subsets of α is a *filter* on α provided

(1) $\varnothing \notin \mathcal{F}$;

(2) if $A \subset B \subset \alpha$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$; and

(3) if $\mathcal{C} \subset \mathcal{F}$ and $|\mathcal{C}| < \omega$ then $\bigcap \mathcal{C} \in \mathcal{F}$.

An *improper filter* on α is a family $\mathcal{F}$ satisfying (2) and (3) such that $\varnothing \in \mathcal{F}$; *i.e.*, $\mathcal{F}$ is the family $\mathcal{F} = \mathcal{P}(\alpha)$.

If $\mathcal{C} \subset \mathcal{P}(\alpha)$ we denote by $\langle \mathcal{C} \rangle$ the (possibly improper) filter on α generated by $\mathcal{C}$; that is,

$$\langle \mathcal{A} \rangle = \{A \subset \alpha\text{: there is finite } \mathcal{B} \subset \mathcal{A} \text{ such that } \bigcap \mathcal{B} \subset A\}.$$

We note in particular that the improper filter on α is not a filter.

DEFINITION. Let $\mathcal{A} \subset \mathcal{P}(\alpha)$. A function $\varepsilon: \mathcal{A} \to \mathcal{P}(\alpha)$ is an *assignment function* if for every $A \in \mathcal{A}$ either $\varepsilon(A) = A$ or $\varepsilon(A) = \alpha\backslash A$.

DEFINITION. Let $\mathcal{F} \subset \mathcal{P}(\alpha)$ and let $\mathcal{G}$ be a filter on α. Then $\mathcal{F}$ is *$\mathcal{G}$-independent* if $\bigcap\{\varepsilon(A): A \in \mathcal{A}\} \cap G \neq \emptyset$ whenever $\mathcal{A}$ is a finite subfamily of $\mathcal{F}$, ε is an assignment function for $\mathcal{A}$, and $G \in \mathcal{G}$.

We note that if $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of subsets of α and $\mathcal{F}$ is $\langle\mathcal{G}\rangle$-independent, then $\langle\mathcal{G}\rangle$ is a filter on α (*i.e.*, $\langle\mathcal{G}\rangle$ is not the improper filter on α).

6.6. THEOREM. *Let $\mathcal{F} \subset \mathcal{P}(\alpha)$, let $\mathcal{G}$ and $\mathcal{H}$ be filters on α such that $\mathcal{F}$ is $\mathcal{G}$-independent and $\mathcal{H}$-independent, and let $f \in \alpha^\alpha$. Then there are $\tilde{\mathcal{F}} \subset \mathcal{F}$ and $A \subset \alpha$ such that*

(1) $|\mathcal{F}\backslash\tilde{\mathcal{F}}| < \omega$, and

(2) $\tilde{\mathcal{F}}$ is $\langle\mathcal{G} \cup \{A\}\rangle$-independent and $\langle\mathcal{H} \cup \{\alpha\backslash f^{-1}(A)\}\rangle$-independent.

PROOF. We fix $F \in \mathcal{F}$ and we consider two cases.

Case 1. $\mathcal{F}\backslash\{F\}$ is $\langle\mathcal{H} \cup \{\alpha\backslash f^{-1}(F)\}\rangle$-independent. We set $A = F$ and $\mathcal{F}' = \mathcal{F}\backslash\{A\}$ and we note that $\mathcal{F}'$ is $\langle\mathcal{G} \cup \{A\}\rangle$-independent. Indeed, if ε is an assignment function on a finite subfamily $\mathcal{A}$ of $\mathcal{F}'$, then since $\varepsilon \cup \{(A,A)\}$ is an assignment function on the finite subfamily $\mathcal{A} \cup \{A\}$ of $\mathcal{F}$ we have

$$\bigcap\{\varepsilon(B): B \in \mathcal{A}\} \cap G \cap A \neq \emptyset$$

whenever $G \in \mathcal{G}$, as required.

Case 2. Case 1 fails. Then there are a finite subfamily $\bar{\mathcal{A}}$ of $\mathcal{F}\backslash\{F\}$, an assignment function $\bar{\varepsilon}$ on $\bar{\mathcal{A}}$, and $\bar{H} \in \mathcal{H}$ such that

$$\bigcap\{\bar{\varepsilon}(B): B \in \bar{\mathcal{A}}\} \cap \bar{H} \subset f^{-1}(F).$$

We set $A = \alpha\backslash F$ and $\tilde{\mathcal{F}} = \mathcal{F}\backslash(\bar{\mathcal{A}} \cup \{F\})$ and we claim that $\tilde{\mathcal{F}}$ is $\langle\mathcal{G} \cup \{A\}\rangle$-independent and $\langle\mathcal{H} \cup \{\alpha\backslash f^{-1}(A)\}\rangle$-independent.

Let ε be an assignment function on a finite subfamily $\mathcal{A}$ of $\tilde{\mathcal{F}}$ and let $G \in \mathcal{G}$ and $H \in \mathcal{H}$. Since $\varepsilon \cup \{(F,A)\}$ is an assignment function on the finite subfamily $\mathcal{A} \cup \{F\}$ of $\mathcal{F}$ we have

$$\bigcap\{\varepsilon(B): B \in \mathcal{A}\} \cap G \cap A \neq \emptyset;$$

and since $\varepsilon \cup \bar{\varepsilon}$ is an assignment function on the finite subfamily $\mathcal{A} \cup \bar{\mathcal{A}}$ of $\mathcal{F}$ we have

$$\emptyset \neq \bigcap\{\varepsilon(B): B \in \mathcal{A}\} \cap \bigcap\{\bar{\varepsilon}(B): B \in \bar{\mathcal{A}}\} \cap (H \cap \bar{H}) \subset f^{-1}(F),$$

and hence,

$$\bigcap\{\varepsilon(B): B \in \mathcal{A}\} \cap H \cap (\alpha\backslash f^{-1}(A)) \neq \emptyset.$$

We note that $\{\alpha\}$ is a filter on α. It follows from Theorem 2.2 for $\alpha \geqslant \omega$ that there is on α an $\{\alpha\}$-independent family $\mathcal{F}$ of subsets of α such that $|\mathcal{F}| = 2^\alpha$.

Indeed let $\{p_\xi : \xi < \alpha\}$ be dense in $\{0,1\}^{2^\alpha}$ and define

$$\mathcal{F} = \{F(\eta): \eta < 2^\alpha\} \subset \mathcal{P}(\alpha)$$

by the rule

$$\begin{aligned} \xi &\in F(\eta) \quad \text{if } p_\xi(\eta) = 1, \\ &\notin F(\eta) \quad \text{if } p_\xi(\eta) = 0. \end{aligned}$$

Then if $\mathcal{Q}$ is a finite subset of $\mathcal{F}$ and ε an assignment function for $\mathcal{Q}$, there is $\xi < \alpha$ such that

$$\begin{aligned} p_\xi(\eta) &= 1 \quad \text{if } F(\eta) \in \mathcal{Q} \text{ and } \varepsilon(F(\eta)) = F(\eta), \\ &= 0 \quad \text{if } F(\eta) \in \mathcal{Q} \text{ and } \varepsilon(F(\eta)) = \alpha \backslash F(\eta); \end{aligned}$$

it follows that $\xi \in \bigcap\{\varepsilon(A): A \in \mathcal{Q}\}$, as required.

6.7. LEMMA. *Let $\alpha \geqslant \omega$ and let $\mathcal{G}$ be the filter*

$$\mathcal{G} = \{G \subset \alpha: |\alpha \backslash G| < \alpha\}.$$

There is $\mathcal{F} \subset \mathcal{P}(\alpha)$ such that $|\mathcal{F}| = 2^\alpha$ and $\mathcal{F}$ is $\mathcal{G}$-independent.

PROOF. We have seen that there is an $\{\alpha\}$-independent family $\mathcal{F}'$ of subsets of α such that $|\mathcal{F}'| = 2^\alpha$. Let f be a function from α to α such that $|f^{-1}(\{\xi\})| = \alpha$ for all $\xi < \alpha$, and define

$$\mathcal{F} = \{f^{-1}(F'): F' \in \mathcal{F}'\}.$$

It is easy to check that $|\bigcap\{\varepsilon(A): A \in \mathcal{Q}\}| = \alpha$ for every finite $\mathcal{Q} \subset \mathcal{F}$ and every assignment function ε for $\mathcal{Q}$, so that $\mathcal{F}$ is $\mathcal{G}$-independent.

We are, finally, prepared for the theorem of Kunen **[Ku1]**, **[Ku2]** that for $\alpha \geqslant \omega$ the Rudin-Keisler order is not linear on $U(\alpha)$.

6.8. THEOREM. *Let $\alpha \geqslant \omega$. There is $S \subset U(\alpha)$ such that $|S| = 2^\alpha$ and the elements of S are pairwise $\leqslant$-incomparable.*

PROOF. Let $\mathcal{G} = \{G \subset \alpha: |\alpha \backslash G| < \alpha\}$. According to 6.7 there is $\mathcal{F} \subset \mathcal{P}(\alpha)$ such that $|\mathcal{F}| = 2^\alpha$ and $\mathcal{F}$ is $\mathcal{G}$-independent.

Let $\{d_\eta: \eta < 2^\alpha\}$ be a well-ordering of the set $\alpha^\alpha \times \{\langle \xi, \zeta\rangle \in \alpha \times \alpha: \xi \neq \zeta\}$; that is, for $\eta < 2^\alpha$ there are $f \in \alpha^\alpha$, and $\xi, \zeta < \alpha$ with $\xi \neq \zeta$, such that $d_\eta = \langle f, \xi, \zeta\rangle$. For $\eta, \xi < 2^\alpha$ we define a family $\mathcal{F}_\eta$ and a filter $\mathcal{G}_{\xi,\eta}$ such that

(i) $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{G}_{\xi,0} = \mathcal{G}$ for $\xi < 2^\alpha$;

(ii) $\mathcal{F}_\eta \supset \mathcal{F}_\zeta$ and $\mathcal{G}_{\xi,\eta} \subset \mathcal{G}_{\xi,\zeta}$ for $\eta < \zeta < 2^\alpha$, $\xi < 2^\alpha$;

(iii) $|\mathcal{F} \backslash \mathcal{F}_\eta| \leqslant \omega + |\eta|$ for $\eta < 2^\alpha$;

(iv) $\mathcal{F}_\eta$ is $\mathcal{G}_{\xi,\eta}$-independent for $\eta, \xi < 2^\alpha$;

(v) $\mathcal{F}_\eta = \bigcap_{\zeta < \eta} \mathcal{F}_\zeta$ and $\mathcal{G}_{\xi,\eta} = \bigcup_{\zeta<\eta} \mathcal{G}_{\xi,\zeta}$ for $\xi < 2^\alpha$ and nonzero limit ordinals $\eta < 2^\alpha$;

(vi) if $d_\eta = \langle f, \xi, \zeta\rangle$ then there is $A_\eta \subset \alpha$ such that

$$A_\eta \in \mathcal{G}_{\xi,\eta+1} \quad \text{and} \quad \alpha\backslash f^{-1}(A_\eta) \in \mathcal{G}_{\zeta,\eta+1}.$$

We proceed by recursion. We define $\mathcal{F}_0$ and $\mathcal{G}_{\xi,0}$ by (i), and $\mathcal{F}_\eta$ and $\mathcal{G}_{\xi,\eta}$ for nonzero limit ordinals $\eta < 2^\alpha$ by (v). To define $\mathcal{F}_{\eta+1}$ and $\mathcal{G}_{\xi,\eta+1}$ for $\eta < 2^\alpha$ we proceed as follows.

Let $d_\eta = \langle f, \xi, \zeta \rangle$. If $\iota < 2^\alpha$ and $\iota \neq \xi$, $\iota \neq \zeta$, then $\mathcal{G}_{\iota,\eta+1} = \mathcal{G}_{\iota,\eta}$. Using Theorem 6.6 we choose $\mathcal{F}_{\eta+1} \subset \mathcal{F}_\eta$ and $A_\eta \subset \alpha$ so that

$|\mathcal{F}_\eta \backslash \mathcal{F}_{\eta+1}| < \omega$, and

$\mathcal{F}_{\eta+1}$ is $\langle \mathcal{G}_{\xi,\eta} \cup \{A_\eta\}\rangle$-independent and $\langle \mathcal{G}_{\zeta,\eta} \cup \{\alpha\backslash f^{-1}(A_\eta)\}\rangle$-independent, and we define

$$\mathcal{G}_{\xi,\eta+1} = \langle \mathcal{G}_{\xi,\eta} \cup \{A_\eta\}\rangle, \quad \mathcal{G}_{\zeta,\eta+1} = \langle \mathcal{G}_{\zeta,\eta} \cup \{\alpha\backslash f^{-1}(A_\eta)\}\rangle.$$

The definitions of $\mathcal{F}_\eta$ and $\mathcal{G}_{\xi,\eta}$ are complete for $\eta, \xi < 2^\alpha$. We note that for $\xi < 2^\alpha$ the set $\bigcup_{\eta<2^\alpha} \mathcal{G}_{\xi,\eta}$ is a filter on α. We choose $p_\xi \in \beta(\alpha)$ such that $\bigcup_{\eta<2^\alpha} \mathcal{G}_{\xi,\eta} \subset p_\xi$ and we note that $p_\xi \in U(\alpha)$ (since $\mathcal{G} \subset p_\xi$).

We set $S = \{p_\xi : \xi < 2^\alpha\}$.

It remains to show that if $\xi, \zeta < 2^\alpha$ and $\xi \neq \zeta$ then p_ξ and p_ζ are $\leqslant$-incomparable. For $f \in \alpha^\alpha$ there is $\eta < 2^\alpha$ such that $d_\eta = \langle f, \xi, \zeta \rangle$ and there is $A_\eta \subset \alpha$ such that

$$A_\eta \in \mathcal{G}_{\xi,\eta+1} \subset p_\xi \quad \text{and} \quad \alpha\backslash f^{-1}(A_\eta) \in \mathcal{G}_{\zeta,\eta+1} \subset p_\zeta;$$

hence $\bar{f}(p_\zeta) \neq p_\xi$ by 3.2(a). It follows that $p_\xi \not\leqslant p_\zeta$ (and, similarly, that $p_\zeta \not\leqslant p_\xi$).

A careless examination of the proof just given might lead one to believe that $\{d_\eta : \eta < 2^\alpha\}$ may be taken to enumerate $\alpha^\alpha \times \alpha \times \alpha$ and that, other details being left unchanged, the resulting ultrafilters satisfy $\bar{f}(p_\xi) \neq p_\zeta$ for all $f \in \alpha^\alpha$ even when $\xi = \zeta$; such a conclusion is of course absurd. Fortunately this suggested modification (and conclusion) are logically inadmissible: in order that $\mathcal{G}_{\xi,\eta+1}$ and $\mathcal{G}_{\zeta,\eta+1}$ be well defined by the relations given it is necessary that $\xi \neq \zeta$.

The following questions have been answered in the affirmative (by Theorems 6.1 and 6.3, respectively) in case $\alpha^+ = 2^\alpha$ or $(2^\alpha)^+ < 2^{2^\alpha}$, but they have apparently not been settled in the general case.

6.9. *Question.* For $\alpha \geqslant \omega$, is there $S \subset U(\alpha)$ such that the elements of S are pairwise $\leqslant$-incomparable and $|S| = 2^{2^\alpha}$? $|S| = (2^\alpha)^+$?

7. **Generalized P-points.** If $\kappa \geqslant \omega$, X is a space and $p \in X$, then p is a P_κ-*point* of X if for every family $\mathcal{U}$ of neighborhoods of p such that $|\mathcal{U}| < \kappa$ there is a neighborhood V of p such that $V \subset \bigcap \mathcal{U}$. We discuss briefly in this section the question of the existence for $\alpha \geqslant \omega$ of P_{ω^+}-points and P_{α^+}-points of $U(\alpha)$.

We note in passing that a nonprincipal ultrafilter p on α is not α^+-complete. Indeed for $\xi < \alpha$ we have $\alpha\backslash\{\xi\} \in p$, while

$$\bigcap_{\xi<\alpha} (\alpha\backslash\{\xi\}) = \varnothing \notin p.$$

The matter at issue is not whether, given $\{A_\xi: \xi < \alpha\} \subset p$, there is $A \in p$ such that $A \subset \bigcap_{\xi<\alpha} A_\xi$; but rather whether, given $\{A_\xi: \xi < \alpha\} \subset p$, there is $A \in p$ such that $\hat{A} \subset \bigcap_{\xi<\alpha} \hat{A}_\xi$.

7.1. LEMMA. *Let $\alpha \geqslant \omega$ and $p \in U(\alpha)$. If p is a selective ultrafilter then p is a P_{α^+}-point of $U(\alpha)$.*

PROOF. Let $\{A_\xi: \xi < \alpha\} \subset p$, say with $A_0 = \alpha$, and define $B = \bigcap_{\xi<\alpha} A_\xi$. We assume in what follows that $B \notin p$ (since otherwise, defining $A = B$, we have $A \in p$ and $\hat{A} \subset \bigcap_{\xi<\alpha} \hat{A}_\xi$, as required). We assume further, replacing A_ξ by $A_\xi \backslash B$, that $\bigcap_{\xi<\alpha} A_\xi = \varnothing$; and we define

$$d_0 = \alpha \backslash A_0, \quad \text{and} \quad d_\xi = \Big(\bigcap_{\zeta<\xi} A_\zeta\Big)\Big\backslash A_\xi \quad \text{for } 0 < \xi < \alpha.$$

Then $\{d_\xi: \xi < \alpha\}$ is a partition of α, and since $d_\xi \notin p$ for $\xi < \alpha$ and p is selective there is $A \in p$ such that $|A \cap d_\xi| \leqslant 1$ for all $\xi < \alpha$. Now for $\xi < \alpha$ we have $A \backslash A_\xi = A \cap (\bigcup_{\zeta \leqslant \xi} d_\zeta)$, so that

$$|A \backslash A_\xi| \leqslant \sum_{\zeta \leqslant \xi} |A \cap d_\zeta| \leqslant |\xi + 1| < \alpha$$

and, hence, $\hat{A} \subset \hat{A}_\xi$, as required.

We note that from 5.5, 4.5, and 7.1 it follows that if $\omega^+ = 2^\omega$ then there is a P_{ω^+}-point of $U(\omega)$. This implication can be established directly, without recourse to the Rudin-Keisler order or to the concept of a selective ultrafilter, and indeed it was established by W. Rudin [**Ru**] en route to the arresting result that, assuming $\omega^+ = 2^\omega$, the space $U(\omega)$ is not homogeneous. The argument is completed by the following appealing sequence of remarks: (1) if every point of $U(\omega)$ is a P_{ω^+}-point, then every G_δ subset of $U(\omega)$ is open; (2) if every G_δ is open, then for every $f \in C(U(\omega), \mathbf{R})$ and $r \in \mathbf{R}$ the set $f^{-1}(\{r\})$ is open; (3) there is $f \in C(U(\omega), \mathbf{R})$ such that $|f[U(\omega)]| \geqslant \omega$; (4) if every point of $U(\omega)$ is a P_{ω^+}-point, there is an infinite cover of the compact space $U(\omega)$ by disjoint (nonempty) open sets. From this it follows, assuming $\omega^+ = 2^\omega$, that $U(\omega)$ contains P_{ω^+}-points and non-P_{ω^+}-points, so that $U(\omega)$ is not homogeneous. This conclusion will be established by quite different methods in §8 below, without the assumption $\omega^+ = 2^\omega$.

As is usual for an infinite cardinal α, we denote by $\mathrm{cf}(\alpha)$ (read: the *cofinality* of α) the least cardinal γ for which there is a set $\{\alpha_\xi: \xi < \gamma\}$ of cardinal numbers such that $\alpha_\xi < \alpha$ for $\xi < \gamma$ and $\sum_{\xi<\gamma} \alpha_\xi = \alpha$.

The following result is from Negrepontis [**N2**, Proposition 4.3].

7.2. LEMMA. *Let $\alpha \geqslant \omega$ and $\kappa \geqslant \omega$, and let p be a P_κ-point of $U(\alpha)$. Then $\mathrm{cf}(\alpha) < \kappa$ or p is a κ-complete ultrafilter.*

PROOF. We assume the result fails, so that $\kappa \leqslant \mathrm{cf}(\alpha)$ and there are $\lambda < \kappa$ and $\{A_\xi: \xi < \lambda\} \subset p$ such that $\bigcap_{\xi<\lambda} A_\xi \notin p$. Replacing A_ξ by $A_\xi \backslash \bigcap_{\zeta<\lambda} A_\zeta$, we assume without loss of generality that $\bigcap_{\xi<\lambda} A_\xi = \varnothing$.

There is $A \in p$ such that $\hat{A} \subset \bigcap_{\xi<\lambda} \hat{A}_\xi$, so that $|A \backslash A_\xi| < \alpha$ for $\xi < \lambda$.

Since

$$A = A\Big\backslash\Big(\bigcap_{\xi<\lambda} A_\xi\Big) = \bigcup_{\xi<\lambda}(A\backslash A_\xi)$$

and $\lambda < \mathrm{cf}(\alpha)$, we have $|A| < \alpha$, a contradiction.

An infinite cardinal α is said to be *Ulam-measurable* if there is a nonprincipal, ω^+-complete ultrafilter on α. It is clear that each uncountable, measurable cardinal (defined as in §4) is Ulam-measurable, and it is a result of Ulam **[U2]**, not difficult to prove (see for example **[CN2**, Theorem 8.31**]**), that every ω^+-complete ultrafilter on a set α is in fact κ-complete for the least measurable cardinal κ; thus the least Ulam-measurable cardinal is a measurable cardinal.

7.3. THEOREM. *Let* $\alpha > \omega$.

(a) *If α is a measurable cardinal then there is a P_{α^+}-point of $U(\alpha)$.*

(b) *If* $\mathrm{cf}(\alpha) > \omega$ *and α is not an Ulam-measurable cardinal then there is no P_{ω^+}-point of $U(\alpha)$.*

PROOF. (a) From Theorem 4.5 there is selective $p \in U(\alpha)$, and the result follows from 7.1.

(b) is the case $\kappa = \omega^+$ of 7.2.

I am indebted to Jerry Vaughan for correspondence (December, 1975) containing his proofs, discovered independently, of Theorem 7.3; for encouraging me to include this theorem in my remarks to you today; and for reminding me that related results, concerning the existence of P_κ-points of certain spaces $\beta(\alpha)\backslash\alpha$, are available in **[SS]** and **[V1]**.

C. THREE PROOFS USING ULTRAFILTERS.

As indicated earlier, the three theorems to be proved here have these features in common: (1) they are recent; (2) obstensibly they do not concern ultrafilters; (3) their proofs (as given here) do use ultrafilters.

8. **A nonhomogeneity theorem.** Our goal in this section is Theorem 8.3 and its corollaries. The basic components of the proof of 8.3 are: the Rudin-Froliḱ (pre-) order on $U(\omega)$, and Kunen's theorem that the Rudin-Keisler order on $U(\omega)$ is not a linear order. A formal, careful definition of the Rudin-Froliḱ order is unnecessary (though the definition is implicit in 8.1 below). Those seeking additional information not required here should consult **[F2]**, **[R2]**, **[Bo]** or **[CN2**, §16**]**.

We say that a subspace D of a space X is *discrete* if for every $p \in D$ there is an open subset U of X such that $U \cap D = \{p\}$; it is not required that D be closed in X.

8.1. LEMMA. *Let $p, q \in U(\omega)$ and suppose there is a discrete, faithfully indexed subset $D = \{x_n: n < \omega\}$ of $U(\omega)$ such that the function $f: \omega \to U(\omega)$ defined by $f(n) = x_n$ $(n < \omega)$ satisfies $\bar{f}(p) = q$. Then $p \preccurlyeq q$.*

PROOF. For $n < \omega$ there is $A_n \subset \omega$ such that $\hat{A}_n \cap D = \{x_n\}$. We assume without loss of generality, replacing A_n by $A_n \backslash \bigcup_{k<n} A_k$ and replacing A_0 by $A_0 \cup (\omega \backslash \bigcup_{n<\omega} A_n)$, that $\{A_n : n < \omega\}$ is a partition of ω. We define $g \in \omega^\omega$ by

$$g(k) = n \quad \text{if } k \in A_n,$$

and we note that $\bar{g}(x_n) = n$ for $n < \omega$. Then $\bar{g} \circ f \in \omega^\omega$, and since $(g \circ f)^-(n) = \bar{g}(x_n) = n$ for $n < \omega$ we have

$$p = (\bar{g} \circ f)^-(p) = \bar{g}(\bar{f}(p)) = \bar{g}(q)$$

and hence $p \preccurlyeq q$.

A subspace D of a space X is said to be *C^*-embedded* (in X) if for every $f \in C(D, [0, 1])$ there is $g \in C(X, [0, 1])$ such that $g | D = f$.

In connection with the following lemma it is well to observe that the union of two (countable) discrete subspaces of a space X need not be discrete, even if each is C^*-embedded. For an example, let $D = \{x_n : n < \omega\}$ be a discrete subspace of $U(\omega)$; then ω and D are disjoint, discrete subsets of $\beta(\omega)$, but $\omega \cup D$ is not discrete.

In 8.2 and 8.3, all closures are taken in the space X: that is, $\mathrm{cl}\, A$ denotes $\mathrm{cl}_X A$.

8.2. LEMMA (FROLÍK [F4], [F5]). *Let X be a space in which each countable, discrete subspace is C^*-embedded, let A and B be countable, discrete subspaces of X, and let $p \in (\mathrm{cl}\, A) \cap (\mathrm{cl}\, B)$. Then*

$$p \in \mathrm{cl}[A \cap \mathrm{cl}\, B] \cup \mathrm{cl}[B \cap \mathrm{cl}\, A].$$

PROOF. Let $C = A \backslash \mathrm{cl}\, B$ and $D = B \backslash \mathrm{cl}\, A$. If the statement fails then $p \in (\mathrm{cl}\, C) \cap (\mathrm{cl}\, D)$. It is clear that C and D are disjoint, countable subsets of X such that $C \cup D$ is discrete, and that the function $f: C \cup D \to [0, 1]$ given by

$$\begin{aligned} f(x) &= 0 \quad \text{if } x \in C, \\ &= 1 \quad \text{if } x \in D \end{aligned}$$

(is continuous and) has no continuous extension to p.

8.3. THEOREM. *Let X be an infinite, compact space in which each countable, discrete subspace is C^*-embedded. Then X is not homogeneous.*

PROOF. Since X is infinite there is an infinite, discrete subspace of X; to ease the exposition we arrange the notation so that $\omega \subset \beta(\omega) \subset X$.

By Theorem 6.8 there are $\preccurlyeq$-incomparable $p, q \in U(\omega)$. We claim that there is no homeomorphism h of X (onto X) such that $h(p) = q$.

Suppose that the claim fails, so that

$$p \in \mathrm{cl}(\omega) \cap \mathrm{cl}(h^{-1}[\omega]), \quad \text{and} \quad q \in \mathrm{cl}(\omega) \cap \mathrm{cl}(h[\omega]);$$

it then follows from 8.2 that

$$p \in \mathrm{cl}(\omega \cap \mathrm{cl}(h^{-1}[\omega])) \cup \mathrm{cl}(h^{-1}[\omega] \cap \mathrm{cl}(\omega)), \quad \text{and}$$

$$q \in \mathrm{cl}(\omega \cap \mathrm{cl}(h[\omega])) \cup \mathrm{cl}(h[\omega] \cap \mathrm{cl}(\omega)).$$

We consider two cases.

Case 1. $q \in \mathrm{cl}(h[\omega] \cap \beta(\omega))$. We define $A = h[\omega] \cap \omega$ and $B = h[\omega] \cap U(\omega)$, so that $q \in \mathrm{cl}\,A \cup \mathrm{cl}\,B$.

If $q \in \mathrm{cl}\,A$ we define $f \in \omega^\omega$ by the rule

$$\begin{aligned} f(n) &= h(n) && \text{if } n \in h^{-1}[A], \\ &= 0 && \text{if } n \in \omega \backslash h^{-1}[A]. \end{aligned}$$

Since $q \in \mathrm{cl}\,A$ we have $p = h^{-1}(q) \in h^{-1}[\mathrm{cl}\,A] = \mathrm{cl}\,h^{-1}[A]$; since f and h agree on $h^{-1}[A]$ we then have $q = h(p) = \bar{f}(p)$ and hence $q \preccurlyeq p$, a contradiction.

If $q \in \mathrm{cl}\,B$ we choose any one-to-one function f from ω to $U(\omega)$ such that

$$f(n) = h(n) \text{ if } n \in h^{-1}[B], \quad \text{and} \quad f[\omega] \text{ is discrete}.$$

Since $q \in \mathrm{cl}\,B$ we have $p = h^{-1}(q) \in h^{-1}[\mathrm{cl}\,B] = \mathrm{cl}\,h^{-1}[B]$; since f and h agree on $h^{-1}[B]$ we then have $q = h(p) = \bar{f}(p)$ and hence (from Lemma 8.1) $p \preccurlyeq q$, a contradiction.

Case 2. Case 1 fails. Then $q \in \mathrm{cl}(\omega \cap \mathrm{cl}\,h[\omega])$, and hence

$$p = h^{-1}(q) \in h^{-1}[\mathrm{cl}(\omega \cap \mathrm{cl}\,h[\omega])] = \mathrm{cl}(h^{-1}[\omega] \cap \beta(\omega)).$$

We define $A = h^{-1}[\omega] \cap \omega$ and $B = h^{-1}[\omega] \cap U(\omega)$, so that $p \in \mathrm{cl}\,A \cup \mathrm{cl}\,B$. An argument similar to that of Case 1 now shows that if $p \in \mathrm{cl}\,A$ then $p \preccurlyeq q$, and if $p \in \mathrm{cl}\,B$ then (from Lemma 8.1) $q \preccurlyeq p$.

8.4. Corollary. *Let $\alpha \geqslant \omega$ and let X be an infinite, closed subspace of $\beta(\alpha)$. Then X is not homogeneous.*

Proof. We assume that $X \subset \beta(\alpha)\backslash\alpha$, since otherwise X contains both isolated and nonisolated points and the conclusion is clear. It is enough to show that every countable, discrete subset $D = \{x_n : n < \omega\}$ of X is C^*-embedded. Let $f \in C(D, [0,1])$, let $\{A_n : n < \omega\}$ be a partition of α such that $x_n \in \hat{A}_n$, and define $g: \alpha \to [0,1]$ by

$$g(\xi) = f(x_n) \quad \text{if } \xi \in A_n.$$

The Stone extension $\bar{g}$ of g satisfies $\bar{g}|D = f$, and we have $f \subset \bar{g}|X \in C(X, [0,1])$, as required.

We note, in particular, from 8.4 that for $\alpha \geqslant \omega$ the spaces $U(\alpha)$ and $\beta(\alpha)\backslash\alpha$ are not homogeneous.

A *cozero-set* in X is the complement of a zero-set–*i.e.*, a set of the form $X\backslash Z$ with $Z \in \mathfrak{Z}(X)$. An *F-space* is a space in which each cozero-set is C^*-embedded. The following lemma is due to Gillman and Henriksen **[GH]**; the proof given here is from Negrepontis **[N1]**.

8.5. LEMMA. *If X is locally compact and σ-compact, then $\beta X \backslash X$ is an F-space.*

PROOF. Let U be a cozero-set in the compact space $\beta X \backslash X$. Then U is an F_σ of $\beta X \backslash X$, so $X \cup U$ is σ-compact and hence normal. The space U is closed in $X \cup U$ (since X is open), so U is C^*-embedded in $X \cup U$, hence in βX, hence in $\beta X \backslash X$.

It is not difficult to prove (see for example [**GJ**, Problem 14N] or [**CN2**, Lemma 16.15(b)]) that every countable subspace of an F-space is C^*-embedded. For our present purposes, an even simpler result is sufficient.

8.6. LEMMA. *Let X be an F-space and D a countable, discrete subspace of X. Then D is C^*-embedded in X.*

PROOF. Let $D = \{x_n : n < \omega\}$, define by recursion a sequence $\{U_n : n < \omega\}$ of pairwise disjoint cozero-sets of X such that $x_n \in U_n$, set $U = \bigcup_{n<\omega} U_n$, and for $f \in C(D, [0,1])$ define $g \colon U \to [0,1]$ by

$$g(x) = f(x_n) \quad \text{if } x \in U_n.$$

Since U is a cozero-set of X and $g \in C(U, [0,1])$ there is $h \in C(X, [0,1])$ such that $h \supset g \supset f$.

8.7. COROLLARY. (a) *No infinite, compact F-space is homogeneous.*

(b) *If X is locally compact, σ-compact and not compact, then $\beta X \backslash X$ is not homogeneous.*

(c) *If X is a space and Z is a nonempty zero-set of βX such that $X \cap Z = \varnothing$, then Z is not homogeneous.*

PROOF. (a) follows from 8.3 and 8.6.

(b) follows from (a) and 8.5, once it is shown that $|\beta X \backslash X| \geqslant \omega$. It is not difficult to find a sequence $\{X_n : n < \omega\}$ of compact subsets of X such that $X_n \subsetneq \operatorname{int} X_{n+1}$ for $n < \omega$, and for every compact $F \subset X$ there is $n < \omega$ such that $F \subset X_n$. Since $\{\operatorname{int} X_{n+1} \backslash X_n : n < \omega\}$ is a locally finite family of subsets of X, a set D chosen so that $|D \cap (\operatorname{int} X_{n+1} \backslash X_n)| = 1$ for $n < \omega$ is closed and C^*-embedded in X. It follows that $\beta D \backslash D \subset \beta X \backslash X$, so that

$$|\beta X \backslash X| \geqslant |\beta D \backslash D| = |U(\omega)| = 2^{2^\omega}.$$

(c) Set $X' = \beta X \backslash Z$. Then X' is locally compact, σ-compact and not compact, and $\beta X' \backslash X' = Z$. Thus (c) follows from (b).

8.8. *Historical notes.* The fact that $U(\omega)$ is not homogeneous was shown, assuming $\omega^+ = 2^\omega$, by W. Rudin [**Ru**] and, without any such assumption, by Frolík [**F2**]. That $U(\alpha)$ is not homogeneous for $\alpha \geqslant \omega$, shown by Negrepontis [**N2**] assuming $\alpha^+ = 2^\alpha$, follows (together with several results in the vein of Corollary 8.4) from this very general result of Frolík [**F5**] (proved with no special set-theoretic assumption): If a closed, nowhere dense subset X of a compact, extremally disconnected space Y contains a homeomorphic copy of Y, then X is not homogeneous. Additional nonhomogeneity results for extremally disconnected, compact spaces, all with hypotheses involving cardinal numbers, are given by Arhangel'skiĭ **[A]**, Efimov **[Ef1]** and Frolík **[F4]**. Corollary 8.7 for X satisfying $\mathrm{w}X \leqslant 2^{\omega}$ is given in **[CN2**, Theorem 16.21].

Frolík **[F3]** (and Isiwata **[I]**, assuming $\omega^{+} = 2^{\omega}$) have shown that if X is not pseudocompact–*i.e.*, if there is an unbounded, continuous, real-valued function on X–then $\beta X \backslash X$ is not homogeneous.

It was Frolík **[F4]** who first noted the importance, for questions about homogeneity, of establishing nonlinearity of (an order stronger than) $\preccurlyeq$ on $U(\omega)$, and Kunen **[Ku1]**, **[Ku2]**, who established nonlinearity.

The first formal announcement of Theorem 8.3 (for extremally disconnected spaces, and with no special hypotheses involving cardinal numbers) using the theorems of Frolík and Kunen is given by Efimov **[Ef2**, editorial note, p. 105].

Additional nonhomogeneity results for some spaces $\beta X \backslash X$ have been given recently by van Douwen **[vD2]** and Bell **[Be]**.

9. **A theorem of Ginsburg and Saks.** We adopt the following notational convention, consistent with usage above. If X and Y are spaces and $f \in C(Y,X)$, we denote by $\bar{f}$ the Stone extension of f from βY to βX. That is, we have $\bar{f} \in C(\beta Y, \beta X)$ and $\bar{f}|Y = f$.

DEFINITION. Let X be a space.

(a) X is *countably compact* if for every $f \in X^{\omega}$ there is $p \in U(\omega)$ such that $\bar{f}(p) \in X$;

(b) if $p \in U(\omega)$, then X is *p-compact* if $\bar{f}(p) \in X$ for every $f \in X^{\omega}$.

We remark that in the presence of our standing convention to the effect that every space is a completely regular, Hausdorff topological space, the definition given above of countable compactness is equivalent to any of the standard definitions. Indeed for $x \in X$ and $f \in X^{\omega}$, it is easy to check that $x \in \bar{f}[U(\omega)]$ if and only if $|\{n < \omega\colon f(n) \in U\}| = \omega$ for every neighborhood U of x in X.

Our point of departure is this result of Scarborough and Stone **[S]**: In order that a product space $\prod_{i \in I} X_i$ be countably compact, it is sufficient that $\prod_{i \in J} X_i$ be countably compact for all $J \subset I$ such that $|J| \leqslant 2^{2^{\mathfrak{c}}}$ (where $\mathfrak{c} = 2^{\omega}$). This result is appealing and a bit surprising: Theorems concerning preservation under the formation of products of properties of compactness type are found rarely and with difficulty, and a "reduction" result of this sort is always pleasing. Nevertheless there arises almost spontaneously the unpleasant feeling that the upper bound given is "too large by one exponential"–*i.e.*, that the restriction $|J| \leqslant 2^{2^{\mathfrak{c}}}$ might properly be weakened to $|J| \leqslant 2^{2^{\omega}}$.

The tool which effects this reduction **[GS]**, **[Sa2]**, introduced for a different purpose in the context of nonstandard analysis by Bernstein **[Bn]**, is the concept of a p-compact space (for $p \in U(\omega)$); we note that Bernstein's p-limit concept (not introduced explicitly in the present treatment) coincides in important special cases with the "producing" relation introduced by Frolík **[F1]**, **F2]**, with one of the orderings considered by Katětov **[K1]**, **[K2]**, and with a definition given independently by Saks **[Sa1**, p. 30].

It should be remarked also that the argument used in 9.2 has proved useful to Ginsburg and Saks in connection with product-space theorems concerning a multitude of topological properties. The simple remarks made in this section should serve as an adequate introduction to **[GS]** and **[Sa2]**, but not as a substitute.

Throughout this section we denote by $\{X_i\colon i \in I\}$ a set of nonempty spaces, for $\varnothing \neq J \subset I$ we write X_J in place of $\prod_{i\in J} X_i$, and we denote by π_J the projection from X_I onto X_J. For $x \in X_I$, we use x_J interchangeably with $\pi_J(x)$.

9.1. LEMMA. *Let $p \in U(\omega)$ and $\{X_i\colon i \in I\}$ a set of p-compact spaces. Then X_I is p-compact. Indeed if $f \in (X_I)^\omega$ and $x \in X_I$ and $(\pi_i \circ f)^-(p) = x_i$, then $\bar{f}(p) = x$.*

PROOF. It is enough to prove the second statement. For $i \in I$ let $\tilde{\pi}_i$ denote the projection from $\prod_{j\in I} \beta X_j$ onto βX_i, and let φ be the continuous function from $\beta(X_I)$ onto $\prod_{i\in I} \beta X_i$ such that $\varphi(q) = q$ for all $q \in X_I$. It is well known and easy to prove (see, for example, **[GJ**, Lemma 7.11] or **[CN2**, Lemma 9.2]) that if $q \in \beta(X_I)$ and $\varphi(q) \in X_I$, then $q \in X_I$.

For $i \in I$ the functions $(\pi_i \circ f)^-$ and $\tilde{\pi}_i \circ \varphi \circ \bar{f}$ are continuous functions from $\beta(\omega)$ to βX_i which agree on ω. It follows that

$$(\tilde{\pi}_i \circ \varphi \circ \bar{f})(p) = (\pi_i \circ f)^-(p) = x_i \in X_i,$$

so that $(\varphi \circ \bar{f})(p) = x \in X_I$ and, hence, $\bar{f}(p) = x$.

9.2. THEOREM. *Let $\{X_i\colon i \in I\}$ be a set of spaces. Then X_I is countably compact if and only if X_J is countably compact for all $J \subset I$ such that* $0 < |J| \leqslant 2^{2^\omega}$.

PROOF. The continuous image of a countably compact space is countably compact, so the "only if" statement follows from the fact that if $\varnothing \neq J \subset I$ then π_J is a continuous function from X_I onto X_J.

Suppose now that X_I is not countably compact and choose $f \in (X_I)^\omega$ such that $\bar{f}[U(\omega)] \cap X_I = \varnothing$. It follows from Lemma 9.1 that for $p \in U(\omega)$ there is $i(p) \in I$ such that $(\pi_{i(p)} \circ f)^-(p) \notin X_{i(p)}$. We choose $J \subset I$ such that $i(p) \in J$ for all $p \in U(\omega)$ and $|J| \leqslant 2^{2^\omega}$, and we define $g = \pi_J \circ f$. Since X_J is countably compact there is $p \in U(\omega)$ such that $\bar{g}(p) \in X_J$ (and hence $\bar{g}(p)_{i(p)} \in X_{i(p)}$). This contradicts the relation

$$\begin{aligned}\bar{g}(p)_{i(p)} &= \pi_{i(p)}(\bar{g}(p)) = \bar{\pi}_{i(p)}((\pi_J \circ f)^-(p)) \\ &= (\pi_{i(p)} \circ f)^-(p) \notin X_{i(p)}.\end{aligned}$$

The theorem just proved, though quite clearly an improvement on the Scarborough-Stone theorem cited earlier, is unfortunately not definitive: It is unknown whether it is a theorem in ZFC that the upper bound 2^{2^ω} is optimal. More specifically, the following question has not been answered.

9.3. *Question.* Is there $\{X_i\colon i \in I\}$ such that $|I| = 2^{2^\omega}$ and X_I is not countably compact and X_J is countably compact whenever $J \subset I$ and $|J| < 2^{2^\omega}$? Whenever $J \subsetneq I$? Is there a space X such that $X^{2^\mathfrak{c}}$ is not countably compact but X^α is countably compact for all $\alpha < 2^{\mathfrak{c}}$ (with $\mathfrak{c} = 2^\omega$)?

I shall give references in 9.8 below to results achieved by several mathematicians which solve or partially solve closely related questions. I have selected for inclusion here just one of these (Corollary 9.7), due to Saks; the proof of 9.7 makes essential use of the concepts we have discussed in §§3 and 4.

9.4. LEMMA. *Let $f \in (\beta(\omega))^\omega$, let $x, p \in U(\omega)$ with p selective and with $\bar{f}(p) = x$, and suppose that if $n < \omega$ then $f(n) \neq x$. Then there are $g \in (\beta(\omega))^\omega$ and $B \in p$ such that g is one-to-one, $g[\omega]$ is discrete, and $g|B = f|B$.*

PROOF. For $n < \omega$ set $d_n = f^{-1}(\{f(n)\})$. Since $\bar{f}(p) = x \neq f(n)$ we have $d_n \notin p$ for $n < \omega$; hence there is $A \in p$ such that $|A \cap d_n| = 1$ for all $n < \omega$. We may therefore assume without loss of generality, replacing f if necessary by a function $h \in (\beta(\omega))^\omega$ such that $h|A = f|A$ and h is one-to-one, that f is one-to-one on ω.

For $k < \omega$ there is $A_k \in f(k)$ such that $x \notin \hat{A}_k$. It is then clear that $\{n < \omega: f(n) \in \hat{A}_k\} \notin p$ for $k < \omega$, since otherwise $x = \bar{f}(p) \in \hat{A}_k$. We set $B_k = \{n < \omega: f(n) \notin \hat{A}_k\}$ for $k < \omega$, so that $B_k \in p$, and we note from Lemma 7.1 that there is $B \in p$ such that $\hat{B} \subset \bigcap_{k<\omega} \hat{B}_k$. We assume without loss of generality, replacing B by an appropriate proper subset of B if necessary, that $|\omega \backslash B| = \omega$. We claim that $f[B]$ is discrete. Indeed for $k < \omega$ the set $\{n \in B: f(n) \in \hat{A}_k\}$ is the finite set $B \backslash B_k$, so for $k \in B$ the open set $U = \hat{A}_k \backslash f[B \backslash B_k]$ satisfies $U \cap f[B] = \{f(k)\}$. For the required function $g \in (\beta(\omega))^\omega$ we choose any one-to-one function g from ω to $\beta(\omega)$ such that $g[\omega]$ is discrete and $g|B = f|B$; that such a function may be defined follows from the fact that $|\omega \backslash B| = \omega$.

For $p \in U(\omega)$, we denote by $F(p)$ the set of all uniform ultrafilters x on ω such that there is $f \in (\beta(\omega))^\omega$ with $\bar{f}(p) = x$ and such that if $n < \omega$ then $f(n) \neq x$. In symbols:

$$F(p) = \{\bar{f}(p) \in U(\omega): f \in (\beta(\omega))^\omega, \quad \bar{f}(p) \notin f[\omega]\}.$$

9.5. LEMMA. *Let p and q be $\sim$-inequivalent selective ultrafilters on ω. Then $F(p) \cap F(q) = \emptyset$.*

PROOF. If the statement fails there are $f, g \in (\beta(\omega))^\omega$ and $x \in U(\omega)$ such that $x \notin f[\omega] \cup g[\omega]$ and $x = \bar{f}(p) = \bar{g}(q)$. From Lemma 9.4 we may suppose without loss of generality that f and g are one-to-one functions and that $f[\omega]$ and $g[\omega]$ are discrete.

Now suppose that $\{n < \omega: g(n) \in \omega\} \in q$, so that $q \sim x$. If $\{n < \omega: f(n) \in \omega\} \in p$, then from $q \sim x$ we have $p \sim q$, a contradiction; and if $\{n < \omega: f(n) \in U(\omega)\} \in p$, then with $C = \{n < \omega: f(n) \in U(\omega)\}$ and with h a one-to-one function from ω to $U(\omega)$ chosen so that $h[\omega]$ is discrete and $h|C = f|C$, we have $\bar{h}(p) = \bar{f}(p) = x \sim q$, hence $p \preccurlyeq q$ from Lemma 8.1, hence $p \sim q$ (since q is $\preccurlyeq$-minimal in $U(\omega)$), a contradiction.

It follows from the preceding paragraph that we may suppose without loss of generality that $f[\omega] \subset U(\omega)$ and $g[\omega] \subset U(\omega)$. Since (from 8.5 and 8.6) every countable, discrete subset of $U(\omega)$ is C^*-embedded, we may by 8.2 suppose further that $x \in \mathrm{cl}[f[\omega] \cap \mathrm{cl}\, g[\omega]]$. Since $\bar{g}$ is a homeomorphism of $\beta(\omega)$ onto $\bar{g}[\beta(\omega)]$ and q is a P_{ω^+}-point of $U(\omega)$, the point x is a P_{ω^+}-point of $\bar{g}[U(\omega)]$ and hence $x \notin \mathrm{cl}[f[\omega] \cap (\mathrm{cl}\, g[\omega] \backslash g[\omega])]$; thus $x \in \mathrm{cl}(f[\omega] \cap g[\omega])$, so $p \in \mathrm{cl}\, f^{-1}(f[\omega] \cap g[\omega])$. We define $h \in \omega^\omega$ by the rule

$$\begin{aligned} h(n) &= m && \text{if } f(n) = g(m) \in f[\omega] \cap g[\omega], \\ &= 0 && \text{if } n \notin f^{-1}(f[\omega] \cap g[\omega]), \end{aligned}$$

and we note that $\bar{h}(p) = q$ and that h is a one-to-one function on $f^{-1}(f[\omega] \cap g[\omega])$, an element of p. It follows from Lemma 3.2(c) that $p \sim q$, a contradiction.

9.6. THEOREM. *Let $\{p_i : i \in I\}$ be a faithfully indexed set of pairwise $\leqslant$-inequivalent selective ultrafilters on ω with $|I| \geqslant 2$, and for $i \in I$ define*

$$X_i = \omega \cup \bigcup \{F(p_j) : j \in I, j \neq i\}.$$

Then

(a) *X_I is not countably compact, and*

(b) *if $\varnothing \neq J \subsetneq I$ then X_J is countably compact.*

PROOF. (a) Define

$$\Delta = \{f \in X_I : \text{there is } x \in \beta(\omega) \text{ such that } f(i) = x \text{ (all } i \in I)\}.$$

We claim that if $f \in \Delta$ and $f(i) = x$ for all $i \in I$ then $x \in \omega$; indeed otherwise there is $i \in I$ such that $x \in F(p_i)$, and then $x \notin X_i$ by Lemma 9.5. Thus Δ is a closed subspace of X_I homeomorphic to the infinite, discrete space ω. It follows that X_I is not countably compact.

(b) Choose $\bar{i} \in I \backslash J$. We claim that each space X_i $(i \in J)$ is $p_{\bar{i}}$-compact. Indeed let $f \in (X_i)^\omega \subset (\beta(\omega))^\omega$: if $\bar{f}(p_{\bar{i}}) \in \omega$, or if there is $n < \omega$ such that $\bar{f}(p_{\bar{i}}) = f(n)$, then $\bar{f}(p_{\bar{i}}) \in X_i$ as required; and if $\bar{f}(p_{\bar{i}}) \in U(\omega) \backslash f[\omega]$, then $\bar{f}(p_{\bar{i}}) \in F(p_{\bar{i}}) \subset X_i$ as required.

We see now that if $\omega^+ = 2^\omega$ then the upper bound 2^{2^ω} of Theorem 9.2 is optimal.

9.7. COROLLARY. *Assume $\omega^+ = 2^\omega$ and let $0 < \alpha \leqslant 2^{2^\omega}$. There is a set $\{X_i : i \in I\}$ of spaces such that*

$|I| = \alpha$,

X_I is not countably compact, and

X_J is countably compact whenever $\varnothing \neq J \subsetneq I$.

PROOF. From Theorem 6.1(a) (and the implication (d) $\Rightarrow$ (c) of Theorem 4.5) there is a faithfully indexed set $\{p_i : i \in I\}$ of pairwise $\leqslant$-inequivalent selective ultrafilters on ω with $|I| = \alpha$. The result now follows from Theorem 9.6.

9.8. *Historical notes.* Theorem 9.2 was proved (for spaces X_i which are pairwise homeomorphic) by Ginsburg and Saks [**GS**, Theorem 2.6]; the more complete, present statement is noted in [C2] and had already been observed independently, together with substantial generalizations, by Saks [Sa2]. The results of 9.5–9.7 are also from Saks [Sa2]. In connection with the statement of 9.7 we note that the assumption $\omega^+ = 2^\omega$ is used only to guarantee the existence of the large family $\{p_i: i \in I\}$; Saks [Sa2] discusses briefly some alternative assumptions sufficient to yield such a family.

Scarborough and Stone [S] raised the question whether every product of sequentially compact spaces is countably compact. This question has been answered in the negative, using a variety of set-theoretic assumptions consistent with ZFC, by Vaughan [V2], by van Douwen [vD1], by Rajagopalan [R], and by Rajagopalan and Woods [RW]. Indeed Kunen [Ku4], assuming $\alpha^+ = 2^\alpha$ for $\omega \leqslant \alpha \leqslant \omega^+$, and Juhász, Nagy and Weiss [JNW] under a weaker assumption, have defined first countable and countably compact (hence sequentially compact) spaces X such that if $p \in U(\omega)$ then X is not p-compact; it is easy to see (as in the last paragraph of the proof of Theorem 9.2) that $X^{2^{\mathfrak{c}}}$ is not countably compact.

10. Glazer's proof of Hindman's theorem. The scene changes, from combinatorial topology to number theory. As with many difficult problems in this branch of mathematics, the statement of the question is quite easily understood. The theorem of Hindman proved below serves to establish the following statement, known for some years as the Graham-Rothschild conjecture.

If the natural numbers are divided into two sets then there is a sequence drawn from one of these sets such that all finite sums of distinct numbers of this sequence remain in the same set.

To prove this, we begin with a simple result from the theory of mobs.

DEFINITION. Let X be a space and $+$ a function from $X \times X$ to X. Then $+$ is *right-continuous* if for all $p \in X$ the function $q \to p + q$ is a continuous function of q.

10.1. LEMMA. *If X is a nonempty compact space and $+$ is an associative, right-continuous function from $X \times X$ to X, then there is a $+$-idempotent in X (i.e., an element p of X such that $p = p + p$).*

PROOF. We define

$$\mathfrak{X} = \{A \subset X: A \neq \varnothing, A \text{ is closed, and } A + A \subset A\},$$

and we note that $\mathfrak{X} \neq \varnothing$ since $X \in \mathfrak{X}$. Ordered by reverse containment, the set $\mathfrak{X}$ satisfies the hypotheses of Zorn's lemma: If $\{A_i: i \in I\}$ is a chain in $\mathfrak{X}$, then with $A = \bigcap_{i \in I} A_i$ we have

$$A + A \subset A_i + A_i \subset A_i \quad \text{for all } i \in I$$

and hence $A + A \subset A$; it follows readily that $A \in \mathfrak{X}$. Hence there is a minimal element of $\mathfrak{X}$. Let B be a minimal element of $\mathfrak{X}$ and choose $p \in B$.

We note that $p + B \neq \varnothing$, that $p + B$ is the image of B under a continuous function and is therefore closed in X, and that

$$(p + B) + (p + B) \subset p + B + B + B \subset p + B;$$

hence $p + B \in \mathfrak{X}$. Since $p + B \subset B$ and B is minimal in $\mathfrak{X}$, we have $p + B = B$ and hence there is $q \in B$ such that $p + q = p$.

Define $C = \{q \in B: p + q = p\}$. Since $C \neq \varnothing$ and C is closed in X and $C + C \subset C$, we have $C \in \mathfrak{X}$; hence $C = B$ and $p + p = p$, as required.

We denote by $\mathbf{N}$ the set of positive integers; that is, $\mathbf{N} = \omega\backslash\{0\}$. Lemma 10.1 will be used in the context of $\beta(\mathbf{N})$. For $A \subset \mathbf{N}$ and $n \in \mathbf{N}$ we set

$$A - n = \{k \in \mathbf{N}: k + n \in A\},$$

and, following Glazer, we define an operation $\dot{+}$ on $\beta(\mathbf{N}) \times \beta(\mathbf{N})$ as follows.

DEFINITION. Let $p, q \in \beta(\mathbf{N})$. Then

$$p \dot{+} q = \{A \subset \mathbf{N}: \{n \in \mathbf{N}: A - n \in p\} \in q\}.$$

10.2. LEMMA (GLAZER). *The function $\dot{+}$ is an associative, right-continuous function from $\beta(\mathbf{N}) \times \beta(\mathbf{N})$ to $\beta(\mathbf{N})$.*

PROOF. We show first that if $p, q \in \beta(\mathbf{N})$ then $p \dot{+} q \in \beta(\mathbf{N})$.

(1) It is clear that $\varnothing \notin p \dot{+} q$.

(2) If $A \in p \dot{+} q$ and $A \subset B \subset \mathbf{N}$, then since $B - n \in p$ whenever $A - n \in p$ we have

$$\{n \in \mathbf{N}: B - n \in p\} \supset \{n \in \mathbf{N}: A - n \in p\} \in q$$

and hence $B \in p \dot{+} q$.

(3) If $A, B \in p \dot{+} q$, then since

$$(A - n) \cap (B - n) = (A \cap B) - n$$

we have

$$\{n: (A \cap B) - n \in p\} = \{n: A - n \in p\} \cap \{n: B - n \in p\} \in q$$

and hence $A \cap B \in p \dot{+} q$.

(4) If $A \subset \mathbf{N}$ and $A \notin p \dot{+} q$, then from $\{n: A - n \notin p\} \in q$ we have $\{n: (\mathbf{N}\backslash A) - n \in p\} \in q$ and hence $\mathbf{N}\backslash A \in p \dot{+} q$.

We show next that $\dot{+}$ is an associative function. Indeed for $p, q, r \in \beta(\mathbf{N})$ and $A \subset \mathbf{N}$ we have

$$\begin{aligned}
A \in (p \dot{+} q) \dot{+} r &\Leftrightarrow \{m: A - m \in p \dot{+} q\} \in r \\
&\Leftrightarrow \{m: \{n: (A - m) - n \in p\} \in q\} \in r \\
&\Leftrightarrow \{m: \{n: A - (m + n) \in p\} \in q\} \in r \\
&\Leftrightarrow \{m: \{k: A - k \in p\} - m \in q\} \in r \\
&\Leftrightarrow \{k: A - k \in p\} \in (q \dot{+} r) \\
&\Leftrightarrow A \in p \dot{+} (q \dot{+} r),
\end{aligned}$$

as required.

We show finally that $\dot{+}$ is right-continuous. According to Lemma 1.2 (and the fact that for $A \subset \mathbf{N}$ and $p \in \beta(\mathbf{N})$ we have $p \in \mathrm{cl}_{\beta(\mathbf{N})} A$ if and only if $A \in p$) it is enough to show that if $p, q \in \beta(\mathbf{N})$ and $A \in p \dot{+} q$ then there is $B \in q$ such that $A \in p \dot{+} q'$ whenever $B \in q'$. To do this, take $B = \{n \in \mathbf{N}: A - n \in p\}$.

We note that the function $\dot{+}$ defined above extends the usual addition function $+$ of $\mathbf{N}$. Indeed, recalling that for $n \in \mathbf{N}$ we have identified n with the ultrafilter $\{A \subset \mathbf{N}: n \in A\}$ on $\mathbf{N}$, we have $\{n\} \in n$ for every $n \in \mathbf{N}$; for $m, k \in \mathbf{N}$ we have $\{m + k\} - n \in m$ if and only if $k = n$, so that $\{m + k\} \in m \dot{+} k$ and hence $m \dot{+} k$ is the ultrafilter (identified with) $m + k$. From these remarks it follows that the ultrafilter p on $\mathbf{N}$ (given by 10.1 and 10.2) such that $p \dot{+} p = p$ is nonprincipal, *i.e.*, $p \in \beta(\mathbf{N})\backslash\mathbf{N}$.

We recall our convention that if A is a set then

$$[A]^{\omega} = \{B \subset A: |B| = \omega\} \quad \text{and} \quad [A]^{<\omega} = \{B \subset A: |B| < \omega\}.$$

For (faithfully indexed) $F = \{k_n: n < m\} \in [\mathbf{N}]^{<\omega}$, we write

$$\sum F = k_0 + k_1 + \cdots + k_{m-1} = \sum_{n<m} k_n.$$

10.3. THEOREM (HINDMAN). *If $\mathbf{N} = \bigcup_{k<n} A_k$ then there are $k < n$ and $B \in [A_k]^{\omega}$ such that $\sum F \in A_k$ whenever $F \in [B]^{<\omega}$.*

PROOF (GLAZER). Define
$\mathscr{Q} = \{A \subset \mathbf{N}$: there is $B \in [A]^{\omega}$ such that $\sum F \in A$ whenever $F \in [B]^{<\omega}\}$. It is enough to show that there is $p \in \beta(\mathbf{N})$ such that $p \subset \mathscr{Q}$; for then if $\mathbf{N} = \bigcup_{k<n} A_k$ there is $k < n$ such that $A_k \in p$.

From 10.1 and 10.2 there is $p \in \beta(\mathbf{N})$ such that $p \dot{+} p = p$. For $A \in p$ we define

$$A^* = \{k \in \mathbf{N}: A - k \in p\}$$

and we note that for every $A \in p$ we have $A \in p \dot{+} p$ and hence $A^* \in p$.

We show $p \subset \mathscr{Q}$. Let $A \in p$, set $A_0 = A$, choose $k_0 \in A_0^* \cap A_0$ and set $A_1 = (A_0 - k_0) \cap A_0$. If $n < \omega$ and if A_n and k_n have been defined with $A_n \in p$ and with $k_n \in A_n^*$, set $A_{n+1} = (A_n - k_n) \cap A_n$ and choose $k_{n+1} \in A_{n+1} \cap A_{n+1}^*$ such that $k_{n+1} > k_n$; that such a choice of k_{n+1} is possible follows by induction from the fact that $A_{n+1} \cap A_{n+1}^* \in p$ and p is nonprincipal on $\mathbf{N}$.

Now define $B = \{k_n: n < \omega\}$. It remains to show that if $F \in [B]^{<\omega}$ then $\sum F \in A_0 = A$. Let $F = \{k_{n(0)}, k_{n(1)}, \ldots, k_{n(m)}\}$ with $m < \omega$ and with $n(i) < n(i+1)$ for $i < m$. We show by downward induction on i that

$$(*) \qquad \sum \{k_{n(j)}: i \leqslant j \leqslant m\} \in A_{n(i)} \quad \text{for } i \leqslant m.$$

For $i = m$, $(*)$ is the statement $k_{n(m)} \in A_{n(m)}$. Now if $i < m$, and if $\sum \{k_{n(j)}: i + 1 \leqslant j \leqslant m\} \in A_{n(i+1)}$, then since $A_{n(i+1)} \subset A_{n(i)+1} \subset A_{n(i)}$

$- k_{n(i)}$ we have

$$\sum \{k_{n(j)}: i + 1 \leqslant j \leqslant m\} \in A_{n(i)} - k_{n(i)}$$

and hence $\sum \{k_{n(j)}: i \leqslant j \leqslant m\} \in A_{n(i)}$. The verification of (*) is complete. Taking $i = 0$ in (*) we have

$$\sum F \in A_{n(0)} \subset A_0 = A,$$

as required.

10.4. *Historical notes.* The Graham-Rothschild conjecture is given in **[GR]**. It was observed some years ago in conversation by Galvin that (in the notation used above) an affirmation of the conjecture would follow from the existence of an ultrafilter p on $\mathbf{N}$ such that $\{n \in \mathbf{N}: A - n \in p\} \in p$ whenever $A \in p$, and Hindman **[H1]** showed, assuming $\omega^+ = 2^\omega$, that the truth of the conjecture is equivalent to the existence of such an ultrafilter.

Hindman established Theorem 10.3, a strong form of the Graham-Rothschild conjecture, in **[H2]**; later a simpler proof, still not so elegant as Glazer's, was discovered by Baumgartner **[Bm]**. According to Glazer the proof above, which is his and is given here with his kind permission, has not been published elsewhere.

Both Hindman and Glazer have observed (independently) in conversation that the technique used here suffices to extend binary operations on X over βX, and to define an idempotent in βX, in a broader setting than is indicated above. For example, using the multiplication $\cdot$ on $\beta(\mathbf{N})$ defined by analogy with the operation $\dot{+}$, one can find $p \in \beta(\mathbf{N})\backslash\mathbf{N}$ such that $p \cdot p = p$. Whether there is $p \in \beta(\mathbf{N})\backslash\mathbf{N}$ such that simultaneously $p \dot{+} p = p$ and $p \cdot p = p$ is, however, apparently unknown.

Lemma 10.1 appears in Wallace **[W1]**, **[W2]**; see also Ellis **[El**, Lemma 2.9**]**.

I am indebted for helpful correspondence and encouragement concerning §10 to the following mathematicians: Haskell Cohen, F. Galvin, S. Glazer, N. Hindman, and R. A. Raimi.

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