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THE GENERALIZED GAMMA FUNCTION, NEW HARDY SPACES, AND REPRESENTATIONS OF HOLOMORPHIC TYPE FOR THE CONFORMAL GROUP[1]

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Operator valued generalizations of the integral formula for the classical gamma function arise in connection with noncompact semisimple, or reductive, Lie groups for which the symmetric space G/K is Hermitian, and they relate to various problems in analysis, group representations, and number theory. In particular, when the holomorphic discrete series for G, constructed originally by Harish-Chandra [3], is realized in terms of the unbounded form of G/K as a Siegel domain, the gamma function plays a decisive role (cf., [1], [2a], [2b], [2d], [6a], [6b]). Moreover, the holomorphic discrete series has an analytic continuation [7], the full extent of which is controlled by the analytic continuation of a normalized version of the gamma function. In general, however, it is only when the gamma function is scalar valued, an occurrence which accounts for but a small part of the holomorphic discrete series, that the full analytic continuation has been determined. In that specialized context, it is known from [6b] that Hardy type Hilbert spaces associated to the various boundary components of G/K appear at the "integer points" in the analytic continuation.

This note announces rather complete solutions to these problems for the conformal group $G = U(2, 2)$. Specifically, we give the entire analytic continuation of the gamma function, the full extent of analytic continuation of the holomorphic discrete series, and we introduce some new vector-valued Hardy spaces.

I. The generalized gamma function. Let $\mathcal{A} = A \times A$ where $A = \mathrm{GL}(2, \mathbf{C})$ and fix a complete set of irreducible holomorphic finite-dimensional representations λ of $\mathcal{A}$ such that $\lambda(a_1, a_2)^* = \lambda(a_1^*, a_2^*)$. Let λ be parametrized by a pair of highest weights $(\sigma_j + 2l_j, \sigma_j)$, $(j = 1, 2)$, where σ_j and $2l_j$ are integers and $l_j \geqslant 0$. Then $\lambda = \lambda(\,\cdot\,; \sigma_1, \sigma_2, \lambda^0)$ where

$$(1) \qquad \lambda(a_1, a_2) = \Delta(a_1)^{\sigma_1}\Delta(a_2)^{\sigma_2}\lambda^0(a_1, a_2)$$

with $\Delta = \det$ and $\lambda^0 = \lambda^0(\,\cdot\,; l_1, l_2)$ a polynomial representation. Let P be

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the cone of positive matrices in A. The *generalized gamma function for G of weight* λ^0 is the holomorphic operator-valued function defined by the absolutely convergent integral

$$\Gamma_{\lambda^0}(\alpha) = \int_P e^{-\mathrm{tr}(r)} \lambda^0(r, \bar{r}) \Delta(r)^{\alpha-2} dr$$

for $\mathrm{Re}(\alpha) > 1$ and elsewhere by analytic continuation.

The following is the main technical theorem. It is proven from the Clebsch-Gordon formula by an involved calculation.

THEOREM 1. *Fix* λ^0. *The values* $\Gamma_{\lambda^0}(\alpha)$ *for* $\mathrm{Re}(\alpha) > 1$ *form a commutative family of normal operators having distinct eigenvalues*

$$\gamma_l^{\lambda^0}(\alpha) = \frac{\Gamma(\alpha + 2l_1)\Gamma(\alpha + 2l_2)\Gamma(\alpha + 2l_1 + 2l_2 + 1)\Gamma(\alpha - 1)}{\Gamma(\alpha + l_1 + l_2 - l)\Gamma(\alpha + l_1 + l_2 + l + 1)}$$

indexed by $l = |l_1 - l_2|, |l_1 - l_2| + 1, \ldots, l_1 + l_2$.

COROLLARY 1. *The function* $\alpha \longrightarrow \Gamma_{\lambda^0}(\alpha)^{-1}$ *extends from* $\mathrm{Re}(\alpha) > 1$ *to an entire function of* α.

We remark that in the special case $l_1 = 0$ or $l_2 = 0$ (cf., [2a]), Γ_{λ^0} is scalar-valued.

Let $t_{\lambda^0}(\alpha) = \mathrm{tr}(\Gamma_{\lambda^0}(\alpha - 2)^{-1}\Gamma_{\lambda^0}(\alpha))$, and define the *normalized gamma function* N_{λ^0} by $N_{\lambda^0}(\alpha) = t_{\lambda^0}(\alpha)\Gamma_{\lambda^0}(\alpha - 2)$.

COROLLARY 2. (i) *The function* $\alpha \longrightarrow N_{\lambda^0}(\alpha)^{-1}$ *is entire.*

(ii) *If either* $l_1 = 0$ *or* $l_2 = 0$ *(so* $N_{\lambda^0}(\alpha)$ *is scalar), then* $N_{\lambda^0}(\alpha)^{-1}$ *is positive for* $\alpha > 1$, *and* $N_{\lambda^0}(1)^{-1} = 0$.

(iii) *If both* $l_1 \neq 0$ *and* $l_2 \neq 0$, *then* $N_{\lambda^0}(\alpha)^{-1}$ *is a positive operator for* $\alpha > 2$, *and* $N_{\lambda^0}(2)^{-1}$ *is nonnegative (one eigenvalue is positive and all others vanish).*

Let $P_1 = \{r \in \mathbf{C}^{2\times 2}: r = r^* \geqslant 0 \text{ and } \Delta(r) = 0\}$, the rank one boundary component of P, and set

$$\widetilde{\Gamma}(\lambda^0) = \int_{P_1} e^{-\mathrm{tr}(r)} \lambda^0(r, \bar{r}) dm(r)$$

where m is relatively A-invariant measure on P_1.

COROLLARY 3. *The operator* $\widetilde{\Gamma}(\lambda^0)$ *is positive. In fact,* $\widetilde{\Gamma}(\lambda^0) = \lim_{\alpha\to 1}(\Gamma_{\lambda^0}(\alpha)/\Gamma(\alpha - 1)) = c(\lambda^0)N_{\lambda^0}(3)$, *where* $c(\lambda^0)$ *is real and positive.*

II. Analytic continuation of the holomorphic discrete series. We realize G/K as the Siegel upper half plane $H = S + iP$ in $\mathbf{C}^{2\times 2}$ where $S = \{x \in \mathbf{C}^{2\times 2}: x = x^*\}$. The *relative holomorphic discrete series* for the universal covering group $\widetilde{G}$ of G consists of representations $T(\cdot, \lambda)$ indexed by λ as in (1) with σ_1, σ_2 real and $\alpha = \sigma_1 + \sigma_2 > 3$. $T(\cdot, \lambda)$ acts in the space $H(\lambda^0, \alpha)$ of holomorphic vector-valued functions F such that

$$\|F\|^2_{\lambda^0,\alpha} = \int_H \|\lambda^0(y, \bar{y})^{1/2} F(x+iy)\|^2 \Delta(y)^{\alpha-4}\, dx dy < \infty.$$

When σ_1 and σ_2 are integers, $T(\cdot, \lambda)$ is a representation of G itself. $H(\lambda^0, \alpha) \neq 0$ if and only if $\alpha > 3$. The reproducing kernel $Q_{\lambda^0,\alpha}$ for $H(\lambda^0, \alpha)$ is calculated (cf., [2d]) to be

$$Q_{\lambda^0,\alpha}(z, w) = \int_P e^{i\operatorname{tr}(z-w^*)r} \lambda^0(r, \bar{r})^{1/2} N_{\lambda^0}(\alpha)^{-1} \lambda^0(r, \bar{r})^{1/2} \Delta(r)^{\alpha-2}\, dr \tag{2}$$

for $z, w \in H$, which can be evaluated as

$$Q_{\lambda^0,\alpha}(z, w) = \Delta(-i(z-w^*))^{-\alpha} \lambda^0(-i(z-w^*), -i(z-w^*)^t)^{-1}. \tag{3}$$

Following [6b], we define the *Wallach set* for G to consist of all (λ^0, α) such that $Q_{\lambda^0,\alpha}$ given by (3) is positive-definite.

THEOREM 2. *Fix any λ^0. Then*: (i) *If $l_1 = 0$ or $l_2 = 0$ (so $N_{\lambda^0}(\alpha)$ is scalar), then (λ^0, α) is in the Wallach set for $\alpha > 1$.* (ii) *If $l_1 \neq 0$ and $l_2 \neq 0$, then (λ^0, α) is in the Wallach set if $\alpha \geqslant 2$.*

The proof is immediate from (2) and Corollary 2. In Case (i), $N_{\lambda^0}(\alpha)^{-1}$ becomes negative for $\alpha < 1$; and in Case (ii), $N_{\lambda^0}(\alpha)^{-1}$ ceases to be a nonnegative operator for $\alpha < 2$. Therefore, *Theorem 2 gives the complete analytic continuation of the holomorphic discrete series.* The special case $\lambda^0 \equiv 1$ (for generic G) is due to Rossi and Vergne [6b], and Case (i) appears in [2a] and [4]. Case (ii) with $2 \leqslant \alpha < 3$ yields new Hilbert spaces $H(\lambda^0, \alpha)$ and new irreducible representations of $\tilde{G}$. The parameter $\alpha = 3$, corresponds to a space $H(\lambda^0, 3)$ which realizes the limit of holomorphic discrete series in the sense of [5].

III. New Hardy spaces. Let $V(\lambda^0)$ be the space of λ^0 and denote by $\mathfrak{H}^2(\lambda^0, P_1)$ the Hilbert space of all holomorphic $F: H \longrightarrow V(\lambda^0)$ such that

$$\|F\|_{\lambda^0,P_1} = \sup_{t \in P} \int_{S+iP_1} \|\lambda^0(y, \bar{y})^{1/2} F(x + i(y+t))\|^2\, dx dm(y) < \infty.$$

THEOREM 3. $\mathfrak{H}^2(\lambda_0, P_1) = H(\lambda^0, 3)$. *In particular,* $\mathfrak{H}^2(\lambda_0, P_1) \neq 0$.

The proof follows from Corollary 3 by a Plancherel theorem argument. For $\lambda^0 = 1$, Theorem 3 is given in [6b]. There, it is also shown that discrete points appear in the Wallach set (for $\lambda^0 \equiv 1$) beyond the range of analytic continuation. The analytic implementation of such spaces for generic λ^0 is an interesting open problem.

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ERRATUM, VOLUME 82

Robert Lee Moore, 1882–1974, by R. L. Wilder

In Volume 82, p. 421, line 14 should read: "analyzes both the form of the group of motions, and the underlying space".