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A TOPOLOGICAL DISK IN A 4-MANIFOLD CAN BE APPROXIMATED BY PIECEWISE LINEAR DISKS

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Several approximation theorems for embeddings of codimension 2 cells are announced here and the proofs are outlined. More detailed proofs will appear elsewhere [5].

1. Introduction. Our main theorem asserts that any topological embedding of a disk (2-cell) in a piecewise linear 4-manifold can be approximated arbitrarily closely by locally flat, piecewise linear embeddings. For codimension 2 cells in general, we do not prove as strong a theorem. If M^n is a piecewise linear (PL) manifold and the topological embedding $D: I^{n-2} \longrightarrow M^n$ has the property that there is some open set $U \subset I^{n-2}$ such that $D|U$ can be ϵ-approximated for every $\epsilon > 0$, then we show that D can be ϵ-approximated for every $\epsilon > 0$. A corollary is that a piecewise linear, codimension 2 cell can be approximated by locally flat $(n-2)$-cells in all dimensions.

If $D: I^{n-2} \longrightarrow M^n$ is the topological embedding, the approximation can be chosen to agree with D on ∂I^{n-2} in both the theorems providing that $D|\partial I^{n-2}$ is PL and $D(\partial I^{n-2}) \subset \operatorname{Int} M$. This can be accomplished simply by pushing the boundary of the approximation to the boundary of D with a small ambient isotopy—using [2] in case $n = 4$ and [3] in case $n \geq 5$. However, if $D(\partial I^{n-2}) \subset \partial M$, the approximation cannot agree with D on the boundary. It is also not possible to replace $\epsilon > 0$ with a function $\epsilon(x) > 0$ with $\epsilon(x) \longrightarrow 0$ as $x \longrightarrow \partial M$. For example, the cone over the trefoil knot in the boundary of E^4_+ cannot be approximated in this way. In fact, the trefoil knot does not bound any locally flat PL disk in E^4_+ [4].

2. Statement of the theorems.

THEOREM 1. *If $D: I^2 \longrightarrow M^4$ is a topological embedding of a disk into a PL 4-manifold, then D can be ϵ-approximated by a locally flat PL embedding $E: I^2 \longrightarrow M^4$ for every $\epsilon > 0$.*

THEOREM 2. *Suppose M^n is a PL n-manifold and $D: I^{n-2} \longrightarrow M^n$ is a topological embedding. If there exists an open set $U \subset I^{n-2}$ such that $D|U$ has*

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the property that $D|U$ can be ϵ-approximated by locally flat PL embeddings for every $\epsilon > 0$, then D has the same property.

COROLLARY. *If $D: I^{n-2} \longrightarrow M^n$ is a PL embedding, then D can be ϵ-approximated by locally flat PL embeddings for every $\epsilon > 0$.*

3. Sketch of the proof of Theorem 1. Begin by choosing $\epsilon' > 0$ and a partition $0 = a_k < a_{k-1} < \cdots < a_1 = 1$ of $I = [0, 1]$ such that if $E: I^2 \longrightarrow M^4$ is any embedding satisfying (1) $E(x \times I) \subset N_{\epsilon'}(D(x \times I))$ for every $x \in I$ and (2) $E(I \times [a_{i-1}, a_i]) \subset N_{\epsilon'}(D(I \times [a_{i-1}, a_i]))$ for each i, then $d(D, E) < \epsilon$. One of the sets $D(x \times I)$ or $E(x \times I)$ will be called a *fiber*. The idea of the proof is to find a sequence of locally flat PL embeddings $E_0, \ldots, E_k$ so that each fiber of E_j is near the corresponding fiber of D for every j and so that E_{j+1} approximates D on one more of the strips $I \times [a_{i-1}, a_i]$ than E_j does.

Let $E: I \times 0 \longrightarrow M$ be a PL approximation of $D|I \times 0$ given by general position. Extend E to $E: I \times I \longrightarrow M$ so that each fiber of E is very short. Consider the homotopy of $E(I \times 1)$ to $D(I \times 1)$ obtained by deforming $E(I \times 1)$ to $D(I \times 0)$ and then moving along fibers of D. Using general position we can make the track of this homotopy miss $E(I \times 0)$. By [1] there is an ambient isotopy which pushes $E(I \times 1)$ near to $D(I \times 1)$, keeps $E(I \times 0)$ fixed, and moves parallel to fibers of D. Let E_0 be the embedding obtained by composing E and this isotopy.

Let U be a neighborhood of $D(I \times [0, a_2])$ and V be a neighborhood of $D(I \times [a_2, a_1])$. To get E_1, we want to pull $E_0(I \times [a_2, a_1])$ into V with an isotopy which moves parallel to fibers of D and keeps $E(I \times [0, a_2])$ near U. Consider the 2-skeleton U^2 of U. By general position, $U^2 \cap E_0(I \times [a_2, a_1])$ is a finite number of points. Associated with each of the points there is a shadow down to $E(I \times a_2)$. We use the techniques of [2] to push these shadows out of V. After this has been done, we can make $E_0(I \times [a_2, a_1]) \cap U^2 = \emptyset$ by simply pushing the a_2 level out over these points staying right on the disk. Now use [1] again to engulf the dual 1-skeleton of U keeping $E_0(I \times 0)$ fixed. The inverse of the engulfing isotopy pulls the image of $I \times [a_2, a_1]$ into V and E_1 is defined as the composition of the maps described. This process is continued inductively.

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