BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, May 1977

THE EXISTENCE OF WAVE OPERATORS IN SCATTERING THEORY

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Communicated by P. Halmos, December 9, 1976

In 1957, Cook [1] gave a sufficient condition for the existence of wave operators. Up to the present time, results obtained by his method are the best, if one is interested merely in proving the existence of the wave operators, and oscillations are not taken into account (cf. [2], [3], [4]). The purpose of this paper is to present a generalization of Cook's criterion which gives stronger results in applications. We exhibit some of these here. Our criterion comes from a new theory of scattering which is much deeper than that required for Cook's theorem.

1. The abstract theory. Let H_0, H be selfadjoint operators on a Hilbert space $\mathcal{H}$, with resolvents $R_0(z), R(z)$, respectively. Define

$$j(z, f, g) = \frac{z - \bar{z}}{2\pi i} (R_0(z)f, [R(z) - R_0(z)] g), \tag{1}$$

and, when it exists,

$$J_I^{\pm}(f, g) = \lim_{0<a\to 0} \int_I j(s \pm ia, f, g)\, ds, \tag{2}$$

where I is an interval. If $I = (a, b)$, we put $\nu_I = \min(-a, b)$.

THEOREM 1. *Suppose f is in the subspace of continuity of H_0 and that for each bounded interval I with ν_I sufficiently large there is a dense set $S_I \subset \mathcal{H}$ such that $J^+(f, g)$ exists for each $g \in S_I$. Assume also that*

$$\limsup_{t\to +\infty} J_I^+ (e^{-itH_0}f, e^{-itH_0}f) \longrightarrow 0 \quad \text{as } \nu_I \longrightarrow \infty \tag{3}$$

for each such I. Then the limit

$$W_+ f = \lim_{t\to\infty} e^{itH}e^{-itH_0}f \tag{4}$$

exists. If the assumptions hold with the plus signs replaced by minus signs, then the limit (4) *exists as $t \longrightarrow -\infty$.*

2. The generalization of Cook's theorem. Again let H_0, H be selfadjoint operators on a Hilbert space $\mathcal{H}$. Suppose there are a Hilbert space $\mathcal{K}$ and oper-

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AMS (MOS) subject classifications (1970). Primary 47A40; Secondary 35J10, 35P25, 81A45.

ators A, B from H to K such that A is H_0-bounded, B is H-bounded and

(5) $$(Hu, v) = (u, H_0 v) + (Bu, Av)_K, \quad u \in D(H), v \in D(H_0).$$

The following is a consequence of Theorem 1.

THEOREM 2. *If* $e^{-itH_0} f \in D(A)$ *for* $t > a$ *and*

(6) $$\int_a^\infty \|A e^{-itH_0} f\| \, dt < \infty,$$

then the limit (4) *exists.*

3. Applications. The following are consequences of Theorem 2. Let H_0 be the selfadjoint operator in $L^2(E^n)$ corresponding to $-\Delta$. If there are constants $m, \gamma > 1$ such that

(7) $$\|\nabla u\|^2 + m^2 \|u\|^2 + \gamma(Vu, u) \geqslant 0, \quad u \in C_0^\infty,$$

then $H_0 + V$ has a selfadjoint forms extension H (cf. [9]).

THEOREM 3. *Assume*

(8) $$(|V| u, u) \leqslant C (\|\nabla u\|^2 + \|u\|^2), \quad u \in C_0^\infty,$$

and that for each $y \in E^n$ *there is an* $a > 0$ *such that*

(9) $$\int_a^\infty t^{-n/2} \left(\int |V(x)| \exp\left\{ -\frac{|x-y|^2}{2(1+t^2)} \right\} dx \right)^{1/2} dt < \infty;$$

then the wave operators

(10) $$W_\pm \Psi = \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \Psi, \quad \Psi \in L^2(E^n),$$

exist.

THEOREM 4. *In addition to* (7) *assume that there is a* p *satisfying* $0 < p \leqslant 2$ *such that*

(11) $$(|V|^p u, u) \leqslant C(\|\Delta u\|^2 + \|u\|^2), \quad u \in C_0^\infty,$$

$$(|V|^{2-p} u, u) \leqslant C(\|\nabla u\|^2 + \|u\|^2), \quad u \in C_0^\infty,$$

and for each $y \in E^n$ *there is an* $a > 0$ *such that*

(12) $$\int_a^\infty t^{-n/2} \left(\int |V(x)|^p \exp\left\{ -\frac{|x-y|^2}{2(1+t^2)} \right\} dx \right)^{1/2} dt < \infty.$$

Then the wave operators (10) *exist.*

COROLLARY 5. *If* (7), (11), *and* (12) *hold and* $(1 + |x|)^\alpha V(x) \in L^p(E^n)$ *for some* $\alpha > (2-n)/p$, $0 < p \leqslant 2$, *then the wave operators* (10) *exist.*

The Dirac operator is given by

$$L_0 = \sum_{j=1}^{3} \alpha_j D_j + m\beta$$

in E^3, where $D_j = \partial/i\partial x_j$ and the α_j, β are 4×4 matrices satisfying certain commutation relations (cf. [10]). If $Q(x)$ is a 4×4 Hermitian matrix function satisfying

(14) $$\sup_x \int_{|x-y|<\delta} |Q(y)|\,|x-y|^{-2}\,dy \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0,$$

then it was shown in [8] that $L_0 + Q$ has a unique selfadjoint extension L such that $D(L) \subset D(|Q|^{1/2})$. Put $l = 1$ if $m \neq 0$ and $l = 0$ if $m = 0$.

THEOREM 6. *If* $(1 + |x|)^\alpha Q(x) \in L^1$ *for some* $\alpha > -l$, *then the wave operators*

(15) $$W_\pm u = \lim_{t \to \pm\infty} e^{itL} e^{-itL_0} u$$

exists on $[L^2(E^3)]^4$.

THEOREM 7. *Assume there is a p satisfying* $0 < p \leqslant 2$ *and an* $\alpha > -l/p$ *such that* $(1 + |x|)^\alpha Q(x) \in L^p$. *If* $q = \max(p, 2-p)$ *and*

(16) $$\sup_x \int_{|x-y|<\delta} |Q(y)|^q |x-y|^{-2}\,dy \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0,$$

then the wave operators (15) *exist.*

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