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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, May 1977

Approximation of functions of several variables and imbedding theorems, by S. M. Nikol'skiĭ, Die Grundlehren der mathematischen Wissenshaften, Band 205, Springer-Verlag, New York, Heidelberg, Berlin, 1975, viii + 420 pp., $46.50.

This book is interesting and important. Although it advertises itself in the introduction as a compact exposition of a number of fundamental questions that have been brought to completion, it is no such thing. In particular, no treatise of 400 plus pages (480 pp. in the original) can be called compact. It is, however, a fascinating description of a mathematical adventure that failed in its main goal but has produced, and continues to produce, much excellent mathematics.

Unfortunately, the editors of this series have given us a seriously flawed English version of the Russian edition that was published in 1969. It has been so badly done that I will take up the question of the translation and its editing in some detail, after I first discuss the text proper.

The book's topic is classes of functions which satisfy various smoothness or differentiability conditions and the determination of which spaces are contained, continuously, in which other spaces or can be mapped, continuously, into such other spaces by appropriate restriction or extension maps. The method used is the method of best approximation by trigonometric polynomials and in the nonperiodic case, by their analogues, the restriction to real space of entire functions of exponential type. We are told in this book how far Nikol'skiĭ and his colleagues got with this topic by the mid 1960's.

The classes of spaces considered are denoted W, H, B, and L spaces. These are decorated in various ways to indicate the amount of smoothness, the L_p-norm in which the smoothness is measured, the space on which the functions are defined, and the "directions" in which the smoothness is measured. Refinements of the notation permit consideration of "anisotropic" spaces where one has different L_p-norms and degrees of smoothness for each direction. Our discussion will mostly be restricted to the isotropic case.

Most of the material in the text is concerned with the case where the functions are defined on all of a Euclidean space $\mathbf{R}^n$. There is some discussion of the situation where the functions are defined on open subsets of $\mathbf{R}^n$ and the problem of extensions to all of $\mathbf{R}^n$. However, this is covered in fuller detail in Nikol'skiĭ's book with Besov and Il'in which has recently appeared in Russian: *Integral representations of functions and imbedding theorems*, Nauka, Moscow, 1975.

I will substantially avoid a detailed discussion of the question of why the spaces are named and denoted as they are, and not some other way (or what is the "correct" way). A few signposts will be included to aid the reader familiar with other traditions than the Sobolev-Nikol'skiĭ way recorded in this book.

Let $L_p = L_p(\mathbf{R}^n)$ be the usual Lebesgue space $(1 \leqslant p \leqslant \infty)$. For periodic functions the norm is taken over a fundamental period. The Sobolev space W_p^l (l a nonnegative integer, $1 \leqslant p \leqslant \infty$) is a space of differentiable functions. $W_p^0 = L_p$ and if $l \geqslant 1$, $f \in W_p^l$ iff $f \in L_p$ and all derivatives (in the distribution sense) of f of order l are also in L_p. W_p^l is a Banach space when normed by the sum of all the L_p-norms of the derivatives of f plus the L_p-norm of f.

The H and B spaces consist of functions satisfying a Lipschitz or Hölder smoothness condition. Let

$$\omega_p(\delta; f) = \sup_{0 \leqslant t \leqslant \delta} \|f(\cdot + t) - 2f(\cdot) + f(\cdot - t)\|_{L_p}.$$

For $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant \infty$, f belongs to the Nikol'skiĭ space H_p^α iff

$$\|f\|_{H_p^\alpha} = \|f\|_{L_p} + \sup_\delta \delta^{-\alpha}\omega_p(\delta; f) < \infty.$$

For $1 \leqslant \theta \leqslant \infty$ the Besov spaces are defined as follows: $B_{p\infty}^\alpha = H_p^\alpha$ and if $1 \leqslant \theta < \infty$, $f \in B_{p\theta}^\alpha$ iff

$$\|f\|_{B_{p\theta}^\alpha} = \|f\|_p + \left[\int (\delta^{-\alpha}\omega_p(\delta; f))^\theta \, d\delta/\delta\right]^{1/\theta} < \infty.$$

An extension is then made to all positive α by a simple trick. Namely, write $\alpha = \bar{\alpha} + \rho$, $\bar{\alpha}$ a nonnegative integer and $0 < \rho \leqslant 1$. One then requires that $f \in L_p$ and that all derivatives of f of order $\bar{\alpha}$ be in $B_{p\theta}^\rho$ and norm the space with the sum of all relevant norms.

Most readers will find these definitions somewhat opaque. They are chosen for efficiency, not clarity. The old fashioned Lipschitz and Hölder functions are really there if one looks hard enough. Let us restrict attention for the moment to periodic functions of one variable. We say that

$$f \in \mathrm{Lip}(\alpha, p) = \Lambda_\alpha^p$$

iff $f \in L_p$ and $\|f(\cdot + h) - f(\cdot)\|_{L_p} \leqslant M|h|^\alpha$, $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant \infty$. These are the Lipschitz spaces as they are usually defined. We also have the "star"-spaces, Λ_*^p, of Zygmund, which are the spaces $H_p^1 = B_{p\infty}^1$ defined above. It is common to designate $\Lambda_\alpha^\infty \equiv \Lambda_\alpha$ and $\Lambda_*^\infty = \Lambda_*$. Then for $0 < \alpha < 1$, H_∞^α is Lip $\alpha = \Lambda_\alpha$, the collection of functions f such that $|f(x) - f(y)| \leqslant M|x - y|^\alpha$ for some $M > 0$. On the other hand H_∞^1 is not Lip $1 = \Lambda_1$, it is Zygmund's class Λ_*. The class Λ_1 turns out to be W_∞^1. Similarly Lip$(1, p)$ is W_p^1 if $1 < p \leqslant \infty$ in the sense that $f \in \mathrm{Lip}(1, p)$ iff it is the integral of a function in L_p; however, Lip(1, 1) properly contains W_1^1 since $f \in \mathrm{Lip}(1, 1)$ iff f is a function of bounded variation, or as it may be suggestively described, the integral of a Borel measure. The last results are due to Hardy and Littlewood. Note here that in some parts of the literature the Besov space $B^{\alpha}_{p\theta}$ is referred to as a Lipschitz space and it is denoted $\Lambda^{p\theta}_{\alpha}$.

Throughout the text Nikol'skiĭ treats the H spaces as a special case instead of proceeding directly with a unified treatment of the B spaces. Perhaps the historical importance of the H spaces as immediate generalizations of Lipschitz spaces was the reason. More likely it is a sentimental attachment to old friends he has studied in much detail.

We are now in a position to describe the main problem with which Nikol'skiĭ and his colleagues were concerned. One of the most fascinating connections between the W spaces and B spaces is the fact that for all α, $B^{\alpha}_{22} = W^{\alpha}_{2}$. The space W^{α}_{2} has been studied by Aronszajn and Calderón (among others) and has been generalized in several ways. One such study was by Slobodeckiĭ who considered the collection B^{α}_{pp}, $1 \leqslant p \leqslant \infty$, which form (in a "natural way") a continuum of spaces. It was a grand hope that a careful analysis of the inclusion relations between the B and W spaces would establish that $B^{l}_{pp} = W^{l}_{p}$ was true for integers l, or lacking that, that at least $B^{l}_{p\theta} = W^{l}_{p}$ for some values of p and θ. The punch line of Nikol'skiĭ's book is that this hope fails. [The faith that this result would be established was so entrenched that for fractional α, W^{α}_{p} was once defined to be what is now designated B^{α}_{pp}. As that faith diminished it was redesignated B^{α}_{p}. The concept of Besov (Lipschitz) spaces $B^{\alpha}_{p\theta}$ developed from these ideas.]

The method of best approximation is quite straightforward in the periodic case. For a function f in L_p one defines $E_n[f]_{L_p} = \inf_{g_n}\|f - g_n\|_{L_p}$, where g_n ranges over all trigonometric polynomials of degree n. The crucial fact in this development is the Bernšteĭn inequality which we state for the one-dimensional case; namely, $\|g_n'\|_{L_p} \leqslant n\|g_n\|_{L_p}$. Bernšteĭn established the $p = \infty$ version of this result. The typical applications of best approximation to the theory of smooth and differentiable functions (which go back to Jackson and Bernšteĭn in the second decade of this century) are that if $0 < \alpha < 1$ then $f \in \mathrm{Lip}(\alpha, p)$ iff $E_n[f] = O(n^{-\alpha})$, and that $f \in W^1_p$ implies that $E_n[f]_{L_p} = O(n^{-1})$.

These results are extended to the nonperiodic case by using "entire functions of exponentional type" in place of the trigonometric polynomials. One defines classes $\mathfrak{M}_{\nu p} \subset L_p$ of the restrictions to $\mathbf{R}^n$ of entire analytic functions of n complex variables such that if $g_\nu \in \mathfrak{M}_{\nu p}$, then the analytic extension of g_ν is of bounded exponential growth, the coefficient ν describing the rate of growth. The crucial consequence of this definition is that if $g_\nu \in \mathfrak{M}_{\nu p}$ then we get a Bernšteĭn inequality, $\|\partial/\partial x_j g_\nu\|_{L_p} \leqslant |\nu|\ \|g_\nu\|_{L_p}$, where $|\nu|$ is a measure of the "size of ν". In other words, the importance of the class of trigonometric polynomials and the related nonperiodic classes $\mathfrak{M}_{\nu p}$ is that the rate at which they can change is strictly controlled (Bernšteĭn's inequality).

For nonperiodic functions we define, as above, $E_\nu[f]_{L_p} = \inf_{g_\nu}\|f - g_\nu\|_{L_p}$ where g_ν ranges over $\mathfrak{M}_{\nu p}$. Restricting attention to the isotropic case [see above; we refer to the case where the smoothness or differentiability is the same in all directions] and letting $\mathfrak{M}_{\nu p}$ represent the so-called "spherical" classes (so that ν is just a positive number), we get that $f \in \mathrm{Lip}(\alpha, p)$ iff $E_\nu[f]_{L_p} = O(\nu^{-\alpha})$, $0 < \alpha < 1$, and that $f \in W^1_p$ implies $E_\nu[f]_{L_p} = O(\nu^{-1})$.

The theorems that describe the relations between the various spaces are called *imbedding theorems*. When Nikol'skiĭ refers to a nonadjectized imbedding theorem, he refers to a continuous inclusion of one space in another, where the p for the L_p-norm is the same for both spaces, but the type of space (H, B, or W) and the smoothness indices (α and θ) or the degree of differentiability (l) can vary. If the p for the L_p-norm can vary, that is called an *imbedding theorem for different metrics*.

There is one more kind of imbedding theorem; namely, *imbedding theorems for different dimensions*. Consider a function f on $\mathbf{R}^n$ that is "smooth enough", and a linear subspace S of dimension m, $0 \leqslant m < n$, which (by a translation) may always be identified with $\mathbf{R}^m$. Is there a "natural" restriction map $\mathfrak{R}$ such that $\mathfrak{R}f = f|_S$ in the case that f is continuous in a neighborhood of S? ($\mathfrak{R}f$ is called a *trace* of f on S.) Given such a map, if f is in some B or W space on $\mathbf{R}^n$, what about $\mathfrak{R}f$ on $\mathbf{R}^m$? Conversely, does there exist a "natural" extension map $\mathfrak{E}$, defined for functions on $\mathbf{R}^m$ to functions on $\mathbf{R}^n$, so that $\mathfrak{R}\mathfrak{E}$ is the identity? Finally, if g defined on $\mathbf{R}^m$ is in a given W or B space, what about $\mathfrak{E}g$ on $\mathbf{R}^n$?

The first two parts. The contents of the book fall into three main parts, the first two of which are easily described in terms of our discussion above. The first part is made up of Chapters 1–3. Chapter 1 covers the usual material (topics from normed linear spaces and measure theory) in 1.1–1.4. In 1.5 a brief review of generalized functions (Schwartz distributions) is given. In 1.5.3–1.5.7 he gives a survey of some aspects of multiplier theory [multipliers are referred to as multiplicators except for the special case of *Marcinkiewicz multipliers*]. In 1.5.9–1.5.10 basic properties of the fractional integration operators are given. They are denoted I_β and are referred to as *operators of Liouville type*. In other parts of the literature they are more commonly known as *Bessel potential operators*. They are defined as a multiplier transform; viz., if $f \to \hat{f}$ denotes the Fourier transform, then $(I_\beta f)\hat{} = (1 + |x|^2)^{-\beta/2}\hat{f}$. These operators are essential to the third part of the book.

Chapter 2 is a brief survey of standard results about trigonometric polynomials. First, it supplies the basic material needed for the study of the periodic classes. Second, it supplies a model for the nonperiodic analogues of trigonometric polynomials, the restrictions to $\mathbf{R}^n$ of entire functions of n complex variables that are of exponential type and are bounded on $\mathbf{R}^n$, which is the subject of Chapter 3.

Chapter 3 is an interesting and valuable exposition and could (with a reduced treatment of the material in Chapters 1–2) have stood by itself as a short monograph. [The reader is warned that there are a number of dangerous typos in this edition. For example in 3.1 on p. 100 we note "$\mathfrak{M}_\nu \subset \mathfrak{M}_{\nu p}$" should read "$\mathfrak{M}_{\nu p} \subset \mathfrak{M}_\nu$", an error that could give the tyro fits.]

The second part of the book is comprised of Chapters 4–7. The main purpose of Chapter 4 is to define W, H and B spaces and give a few of their important properties. In 4.3 the spaces are defined.

In 4.4 some a priori estimates are given; i.e., estimates of L_p norms of "intermediate" derivatives of functions are given in terms of the norm of the function and norms of higher order derivatives. In 4.7 the spaces are shown to be complete. 4.6 and 4.8 are given to showing how one obtains estimates of difference increments from the behaviour of the derivative and, conversely, estimates of the derivative from the behaviour of difference increments.

Chapter 5 is the longest chapter in the book. It is also the central chapter. It deals with the application of the method of best approximation to the H, B, and W spaces and to showing that a big bunch of different norms for these spaces are equivalent. The major conclusion is that the Besov (B) spaces (and so, in particular, the Nikol'skiĭ (H) spaces) can be characterized by the extent to which they can be approximated by entire functions of exponential type, but while the Sobolev (W) spaces imply a degree of approximation, they cannot be so characterized.

To satisfy any reader who is now wondering why my remarks are limited to the case of isotropic spaces defined on all of $\mathbf{R}^n$, I will exhibit a Besov space fully decorated: $B^{\alpha}_{p\theta}(\Omega)$. Ω is an open subset of ξ. $\xi = \mathbf{R}^m \times \xi$, $\xi' \subset \mathbf{R}^{n-m}$, $1 \leqslant m \leqslant n$, $p = (p_1, \ldots, p_m)$, $\theta = (\theta_1, \ldots, \theta_m)$, $\alpha = (\alpha_1, \ldots, \alpha_m)$, $1 \leqslant p_k$, $\theta_k \leqslant \infty$, $\alpha_k > 0$, so that f is defined on $\Omega \subset \mathbf{R}^n$ and has smoothness in the direction of x_k $(k = 1, \ldots, n)$ measured in L_{p_k} norm and in an amount determined by α_k and θ_k. Furthermore, the spaces can be extended to permit $\alpha_k \leqslant 0$ in which case the objects in the spaces must be viewed as generalized functions (i.e., Schwartz distributions).

Chapter 6 is concerned with connections among the various B and W spaces. In Chapter 7 the connections (i.e., imbedding theorems) are shown to be sharp; in the language of Nikol'skiĭ, unimprovable. To illustrate the results obtained, we ask when does one have the imbedding $B^{\alpha}_{p\theta}(\mathbf{R}^n) \to B^{\beta}_{q\lambda}(\mathbf{R}^m)$ where the imbedding is an inclusion map if $m = n$, a restriction map if $n > m$, an extension map if $n < m$. Then the following conditions are necessary and sufficient: (1) $q \geqslant p$; (2) $\alpha - n/p > \beta - m/q$, θ arbitrary; or (2') $\alpha - n/p = \beta - m/q$ and $\theta \leqslant \lambda$. For the periodic case condition (1) is replaced with (1*) $\alpha \geqslant \beta$. For the W classes if $m = n$ (ignoring, of course, reference to θ and λ and noticing that α and β are integers) the conditions are also necessary and sufficient. Anisotropic versions of these results are given.

The connections between the B and W spaces in these two chapters are restricted to rather elementary relations, such as $H_p^{l+\varepsilon} \to W_p^l \to H_p^l$ for $\varepsilon > 0$.

He also gives several *theorems on compactness*. An example follows:

Let $\{f_k\}$ be a sequence of functions that is bounded in a $B^{\alpha}_{p\theta}$ space. Then there exists $f \in B^{\alpha}_{p\theta}$, with norm bounded by the bound of the sequence, and a subsequence $\{f_{k_l}\}$ such that $\{f_{k_l}\} \to f$ in H_p^{β} for all $\beta < \alpha$.

More introduction. Before continuing with a description of the last part of the book, I will stop to describe two notions that are developed in Chapter 8 and are of fundamental importance. The first is related to the fractional integration operators I_α that were introduced in 1.5.9 (see above). The idea behind these operators is quite reasonable. One starts with a notion of a fractional integral for periodic functions of one variable and then generalizes appropriately. Consider the operators $f \to f_\alpha$ defined as follows: If $f(x) \sim \Sigma c_\nu e^{i\nu x}$ and $f_\alpha(x) \sim \Sigma' c_\nu |\nu|^{-\alpha} e^{i\nu x}$, where "$\Sigma'$" indicates summation without the $\nu = 0$ terms. For $0 < \alpha < 1$ this mapping is realized by convolution with an integrable kernel which behaves like $|x|^{\alpha-1}$ at the origin. This *fractional integration* operator is extended in the nonperiodic case as a multiplier operator, using the multiplies $|x|^{-\alpha}$. This operator is usually referred to as a *Riesz potential* operator.

For $0 < \alpha < n$ these Riesz operators can be realized by convolution, in the principal value sense, with a multiple of $|x|^{\alpha-n}$. Since this kernel does "not behave well at infinity" it is usually fixed up by replacing it with a "nicer" kernel. An acceptable alternative is the operator that corresponds to the multiplier $(1 + |x|^2)^{-\alpha/2}$. These operators are referred to as *Bessel potential operators*, and are referred to by Nikol'skiĭ as operators of Liouville type in honor of the fact that Liouville was one of the first to define fractional integrals (along with Riemann), although the definition above (for $f \to f_\alpha$) is closer to the notion developed by Weyl. The operator $(I_\alpha f)\hat{} = (1 + |x|^2)^{-\alpha/2}\hat{f}$ is defined for all complex α and all distributions f. Furthermore, for $\alpha > 0$, $I_\alpha f = G_\alpha * f$, where G_α is a positive integrable function that decreases at an exponential rate at infinity.

The next idea is the representation of a "nice" function by means of its de la Vallée Poussin sum. If we start with the idea that an L_p-function is "very nice", then a distribution, f, is "nice enough" or *regular in the sense of* L_p if $I_\alpha f \in L_p$ for some α.

One of the truly fascinating facts about best approximation by trigonometric polynomials is that there is a simple way to construct a polynomial of degree $2n$ that does as well as the best approximation of degree n. One just uses the so-called delayed arithmetic means. Thus, if $s_n = s_n(x; f)$ is the nth partial sum of the Fourier series of f, one defines $\sigma_n = (s_{n+1} + \cdots + s_{2n})/n$, and with a little work one finds that $E_{2n}[f]_{L_p} \leqslant \|f - \sigma_n\|_{L_p} \leqslant ME_n[f]_{L_p}$ if $f \in L_p$. By an obvious extension one can define $\sigma_n = \sigma_n(x; f)$ for any regular f by: $\sigma_n(x; f) = I_{-\beta}\sigma_n(x; I_\beta f)$ for sufficiently large β. Then one obtains the de la Vallée Poussin sum of f: $f = \sigma_1 + \sum_{k=1}^{\infty}(\sigma_{2^k} - \sigma_{2^{k-1}})$, which converges to f in the distribution sense.

To see the power of this construction note that we now have some very simple characterizations of $B^\alpha_{\infty\infty}$ ($\equiv$ Lip α, if $0 < \alpha < 1$); namely, $\|f - \sigma_n\|_\infty = O(n^{-\alpha})$ or $\|\sigma_{2^k} - \sigma_{2^{k-1}}\|_\infty = O(2^{-k\alpha})$.

The third part. The plan of Chapter 8 is as follows: In 8.1 the functions G_β ($\hat{G}_\beta = (1 + |x|^2)^{-\beta/2}$), $\beta > 0$ are described. These are the so-called Bessel-MacDonald functions. In 8.2 it is shown that I_l maps L_p ($1 < p < \infty$) isomorphically onto W_p^l so that the Sobolev spaces (for fixed p) are all isomorphic to L_p and, a fortiori, to each other. In 8.3 more properties of the map I_β are described, and in 8.4 comes the crucial fact that if $f \in L_p$ and $\beta > 0$, then $E_\nu[I_\beta f]_{L_p} \leqslant C_\beta|\nu|^{-\beta}E_\nu[f]_{L_p}$. In 8.5 and 8.6 the notion of the de la Vallée Poussin sum is generalized to apply to regular functions defined on $\mathbf{R}^n$, periodic and nonperiodic.

In 8.7 a technical lemma on the behaviour of $I_{-\beta}$ ($\beta > 0$) is established so that in 8.8 and 8.9 one can establish some facts about Besov spaces. First, whether or not $f \in B^\alpha_{p\theta}$ is completely determined by its de la Vallée Poussin sum in the sense that

$$\|f\|_{L_p} + \left[\sum_{k=1}^{\infty} 2^{k\alpha\theta}\|\sigma_{2^k}(f) - \sigma_{2^{k-1}}(f)\|^\theta_{L_p}\right]^{1/\theta}$$

is a norm for $B^\alpha_{p\theta}$. Second, I_β maps $B^\alpha_{p\theta}$ isomorphically onto $B^{\alpha+\beta}_{p\theta}$, which is not very hard to see once one has the de la Vallée Poussin characterization of the Besov spaces. Using this approach, the study of the inclusion relations among the Besov spaces and their relation to the Sobolev spaces is much simplified, at least for the isotropic case.

In Chapter 9 the discrete collection of Sobolev spaces $\{W_p^l\}$ is dropped in favor of a more rational and "continuous" collection $L_p^\alpha = \{f\colon f = I_\alpha g, g \in L_p\}$. One sets $\|f\|_{L_p^\alpha} = \|g\|_{L_p}$. From our comments above we see that $L_p^l = W_p^l$ and also that we have a continuum of spaces L_p^α to compare with $B_{p\theta}^\alpha$. Similarly one can construct an "anisotropic" kernel of the Bessel-MacDonald variety and construct $L_p^\alpha(\mathbf{R}^n)$ spaces where α is an n-vector, and one can then obtain the so-called *integral representations of anisotropic Liouville classes*. Nikol'skiĭ refers to the L_p^α space as a *Liouville space*. It is also known as a *Lebesgue space* and as *a Bessel potential space*.

In the isotropic case the following results are obtained: If $p = 1$ or ∞ then $B_{p1}^\alpha \subset L_p^\alpha \subset B_{p\infty}^\alpha$. If $1 < p < \infty$, $r = \min[p, 2]$, $s = \max[p, 2]$, then $B_{pr}^\alpha \subset L_p^\alpha \subset B_{ps}^\alpha$. In particular, $B_{22}^\alpha = L_2^\alpha$, and if $p \neq 2$, each of the inclusions is proper and unimprovable. As a consequence, if $p \neq 2$, $B_{p\theta}^\alpha \neq L_p^\alpha$ for any θ.

The reviewer is delighted to have the chance to point out that "nonimprovability" of these inclusions is the subject of the ultimate section of Nikol'skiĭ's book and is done there in rather incomplete form; whereas it was done in this reviewer's dissertation (1962) by means of some quite elementary examples.

The theorems on imbedding involving a change of dimension are given for L_p^α spaces. They take the following form: If $1 < p \leqslant \infty$, $\mathfrak{R}\colon L_p^\alpha(\mathbf{R}^n) \to B_{pp}^{\alpha-1/p}(\mathbf{R}^{n-1})$ and if $1 \leqslant p < \infty$, $\mathfrak{E}\colon B_{pp}^\alpha(\mathbf{R}^n) \to L_p^{\alpha+1/p}(\mathbf{R}^{n+1})$, so that if $1 < p < \infty$ the L_p^α spaces and the B_{pp}^α spaces have exactly the same traces, a beautiful and unexpected result which is as close as we get to confirming the hypothesis that they are the same space.

Some summary comments on the contents. The contents of the book may be summarized as follows: The story starts with the Sobolev spaces, W_p^l (spaces of differentiable functions) and the Nikol'skiĭ spaces, H_p^α (spaces of smooth functions). These spaces are to be studied by means of the method of best approximation. The H spaces are extended to the Besov spaces, B_p^α (a wider class of smooth functions) with a pause to consider the particular spaces B_{pp}^α which were studied by Slobodeckiĭ. The B_{pp}^α spaces are also denoted B_p^α and for fractional values of α were originally denoted W_p^α. This notation embodied the expectation that W_p^l would be equal to B_{pp}^l for integer values of l, a result which does hold for $p = 2$ (which is, incidentally, a trivial result which follows from examining the Fourier transforms of functions in the two classes). After much effort and detailing of results on best approximation and on functions of exponential type (results of independent interest) one obtains the embedding theorems in Chapters 6–7.

It is clear on first principles that special consideration of the H spaces is not necessary or useful. In view of the results in Chapter 8 it is clear that if a compact exposition of results about Besov and Sobolev spaces is desired within the framework of results in this book then the Besov spaces should be defined by means of de la Vallée Poussin sums and the Sobolev spaces by means of the Bessel potentials I_β. As the author indicates in his introduction and in various remarks throughout the book, he agrees with this view on the one hand, but not with the hand that wrote the book. Do not be deceived that Nikol'skiĭ has taken this approach without knowing what he is doing. He has done so explicitly and on what seem to be sentimental grounds. For example, in 4.3 he discusses why he does not drop the W_p^l notation for the Sobolev spaces (restricting l to the nonnegative integers) for the more appropriate Liouville classes (which he finally does anyway in Chapter 9 after 300 pages have passed).

He explains: " . . . I have not done this in this book, because I feared to be like a person who thinks it appropriate to rename a street and renames it without asking the opinion of those inhabiting the street."

A most puzzling aspect of this book is why the author has chosen to almost totally ignore the approach of studying the relations between the B and L spaces by a method related to the method of best approximation, but one that avoids most of its complications. In this approach one takes a particular smooth regularization of the distribution and uses the behaviour of that regularization or its derivative. It is by this method that one finally obtains the inclusion relations between $B^{\alpha}_{p\theta}$ and L^{α}_{p} in their sharpest form. This development was carried out simultaneously and independently in 1962 by Lizorkin (using metaharmonic functions) and by the reviewer (using harmonic functions) in his dissertation, results which Nikol'skiĭ knew of, and refers to incidentally. Similarly he has ignored the contributions of the theory of interpolation of linear operators. The spaces in this book can be realized as intermediate spaces in the sense of interpolation theory.

The translated edition. The translated edition as it has been presented to us by the editors of this series is grotesque. This is a shame since Nikol'skiĭ's book deserved a much better treatment, and the translator deserved better treatment from his editors. I am of the opinion that what we have here is a passable first draft which was then left essentially unedited, particularly with respect to its technical contents. The result is that we have a translation which appears to be the product of persons who know something of Russian, clearly have significant difficulties with mathematical English, know very little about the general mathematical field (harmonic analysis) in which the subject of the book resides and are functional illiterates in the specific subject at hand. Nikol'skiĭ must take some of the responsibility since in a "Translator's Note" and an "Author's Preface to the English Edition" he is given, and he accepts, credit for helping to achieve a correct translation. The ultimate responsibility rests with the editors of the series.

I will not take the space to fully document these complaints but I will give a few examples. One example is that throughout the text, the word "notations" is used for "notation", so that the text reads as if it was written with an accent. Another is the use of the arcane and obscure "multiplicator" for "multiplier". The best that can be said for "multiplicator" is that it is a transliteration of the word in Russian and it sounds very much like its equivalent in French.

These are, of course, mere annoyances. But (p. 96) to translate *pusto* as "empty" when "trivial" is meant shows that the translation was mechanical and that meaning was not a major consideration. On p. 51 we are to consider functions " . . . with Fourier coefficients $c_{\mathbf{k}}$, not equal to zero unless $\mathbf{k} \geqslant 0$ . . . ". Anyone with a passing knowledge of Fourier series would recognize this as a very strange class of functions and context alone would demand a reinterpretation as functions " . . . with Fourier coefficients $c_{\mathbf{k}}$, equal to zero unless $\mathbf{k} \geqslant 0$ . . . ". The phrase in question can be translated as it was given, but it makes no sense. The difficulty lies with correctly interpreting the colloquial expression "*ne* · · · *čto dlya*."

Most of the typographic expressions in the original have been carried over intact. Since a substantial number of new typographical errors have been introduced, the text contains a large number of such errors. Two examples of errors being carried over: The use of "b" for "B" in the left hand side of 4.3.4(5). The footnote in 9.3 to a "Remark to 9.3" which does not exist. Why didn't anyone check to see if it was there?

There are entire paragraphs for which prizes could be offered for the best guess as to what the original meant (no peeking at the original permitted). On p. 381 in the Remarks to Chapter 4, the one sentence paragraph following the statement of Theorem 1 would make a good entry for such a contest.

A major problem for an author of a book, where the relevant literature covers various language groups, is to refer to a literature which is available to, and readily understandable by, his intended audience. In the original text, Nikol'skiĭ did an excellent job for his Russian audience, in this respect. It is, of course, an elementary responsibility in a translation from Russian to English to reverse this process. If an original article or book is in English, refer to it, not a Russian translation. Important material originally in Russian but available in English should be referred to in the English edition. It is also common practice to refer to German or French editions when English editions are not available.

We are led to believe, by a translator's footnote on p. 377, that this will be done. He tells us: "In this edition we refer where possible to English or German versions of these books." This is stated in immediate reference to books on generalized functions (distributions) by Gel'fand and Šilov, Vladimirov, and Halperin. All three references are to Russian editions. All three are available in English; respectively, Academic Press, 1964, MIT Press, 1966 and University of Toronto Press, 1952. The last of these was originally in English. Yet our translator tells us that he is not able to find the reference! The very next reference on that page is to the " . . . book of Hörmander [**1**], where far-reaching results on multiplicators are obtained . . . ". The reference is to a translation in Russian, and we are told that the translator does not have the original reference. Our only hint is the title: *Estimates for operators invariant relative to a displacement*. It is not difficult to trace this back to *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140.

The number of such examples is large, but to terminate this list of absurdities we find that the translator with the help of his editors was unable to locate editions in English of Watson's treatise on the theory of Bessel functions or either of Zygmund's books on trigonometric series. Perhaps it was an attempt at humor (the ultimate Polish joke) to refer to the first edition of Zygmund's book as the "old edition in Russian" when it was actually published in Warsaw in an English edition.

As I said earlier, Nikol'skiĭ's book deserved better. It is a fine retelling of an important piece of mathematical research. While written from a rather parochial point of view it does tell an important story and reports on some most interesting mathematics. The editors have chosen, however, to present it in a manner that imposes great difficulties for the reader.

It should be handled with care.

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