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BOOK REVIEWS

Graph theory with applications, by J.A. Bondy and U.S.R. Murty, American Elsevier Publishing Company, Inc., New York, New York, x + 164 pp., $19.50.

When I first entered the world of Mathematics, I became aware of a strange and little-regarded sect of "Graph Theorists", inhabiting a shadowy borderland known to the rest of the community as the "slums of Topology". What changes there have been in a few short years! That shadowy borderland has become a thriving metropolis. International conferences on Graph Theory occur with almost embarrassing frequency. Journals on Graph Theory abound: I once counted the Editorial Offices of three of them in one of the mathematical departments of one of the Universities of one of the smaller cities of Canada. Any connection with Topology is likely to be firmly repudiated as soon as noted.

I became aware of the burgeoning of Graph Theory when I studied the 1940 paper of Brooks, Smith, Stone and Tutte in the Duke Mathematical Journal, ostensibly on squared rectangles. They wrote of trees and Kirchhoff's Laws, of 3-connection and planarity, of duality and symmetry, of determinantal identities and coprime integers, – all in the Quest of the Perfect Square.

I invariably recommend that paper to my students. "Go to it", I say, "you will

'Find tongues in trees, books in the running Brooks,
Sermons in Stones, and good in every thing' ". [8].

Sometimes they complain that the paper is difficult, being written in the terse and condensed style befitting an era of paper shortage. But from now on I shall have a reply to this. I shall refer them to the book of Bondy and Murty, saying that it makes an excellent introduction to the Brooks paper, and to many other manifestations of Graph Theory. Bondy and Murty may be a little short on determinantal identities, – not that that worries me, – but they do conclude with a marvelous Perfect Square.

Graph Theory began with a tour of Königsberg and the problem of Eulerian paths [4]. This part of the discipline was supplemented a century later with a discussion of the superficially analogous Hamiltonian paths. There is a chapter on all this in Bondy and Murty, so up-to-date as to include what I believe to be the first published account of the Horton graph. This is the first example of a non-Hamiltonian 3-connected biparite graph. I wish I could conscientiously put less praise into this review; I have not forgiven the authors for misspelling my name.

A second source of Graph Theory is to be found in the work of Cayley on trees and of Kirchhoff on trees and electric currents. Kirchhoff really started something, both in Physics and Mathematics. For the latter discipline he introduced the subject of Algebraic Graph Theory.

"Down in the forest something stirred"
'Twas neither reptile, beast nor bird
But Father Kirchhoff, full of glee,
Plucking currents off a tree.

You can read about these things in Bondy and Murty. They have a chapter on trees and another, called "The cycle space and bond space" on Kirchhoff's Laws.

I approach Algebraic Graph Theory with some misgivings. I like to make diagrams of my graphs and to find my proofs by tracing or modifying the graphic structure. I joy in the work of Hassler Whitney as he builds up a Hamiltonian circuit in a planar triangulation (without separating triangles). His proof is long, but each step is straightforwardly graph-theoretical and easy to explain by way of a diagram. I do not object to his use of mathematical induction; that is only a concise way of indicating the successive steps in the proof. But contrast even some of the work of Brooks et al.! They have a theorem saying that, under what appear to be common-sense conditions, the replacement of part of a graph by its mirror image makes no difference to the number of spanning trees. To this day no one has proved this result by diagrammatic tinkering. The proof depends on determinantal identities. "Period", as the Americans say. This state of affairs fills me with dismay. I feel that my Graph Theory is being invaded and absorbed by something alien.

In such a mood I find comfort in Bondy and Murty. I turn, for example, to the Chapter "Planar Graphs". There, in the proof of Kuratowski's Theorem I encounter real Graph Theory, with beautiful diagrams explaining what bridges across a circuit are, and how they can overlap. But, perhaps unfortunately, I have further knowledge. There is another theorem, not discussed in the book, that is claimed to be a generalization of Kuratowski's Theorem. It states a "condition for a binary matroid to be graphic". There is still talk of bridges, but they have become vector spaces modulo 2. Circuits have become vectors, and when bridges "overlap" they contrive to do so in an algebraic manner. The proof does indeed exhibit a graph in its last paragraph, but it is a piece of algebra for all of that. So is Kuratowski's Theorem really something in Algebra, and does it appear in its graph-theoretical dress only at parties for the children? [7]. Well, at least we have liberated it from Topology.

I have worked on vertex-colourings and edge-colourings [1], [2], [3], and I rejoice to find these things so clearly discussed in Chapters 6 and 8. I approve of the proofs of Brooks' Theorem and Vizing's Theorem: there again we have genuine Graph Theory. But in these subjects too we find with wider reading that there are algebraic formulations in which a graph dissolves into a vector space and a colouring becomes a vector with no zero coefficients. But this time I think the algebraic formulations have not been notably successful in advancing knowledge. (I claim chromatic polynomials for Graph Theory.) Even the recently announced proof of the Four Colour Theorem, vast in volume, seems to me to be all Graph Theory.

Where is Graph Theory going? Sometimes I fear that it is going down in a sea of Algebra. We have noted the tendency already in the Brooks' paper. As part of my preparatory work for this review I looked again at some of the later work of one of Brooks' coauthors, and found more of the same. Does he want to prove that one Hamiltonian circuit implies at least 3 in a trivalent graph? He does it by a piece of algebra. Does he want a condition for a graph to have a perfect matching? (See Chapter 5.) He extracts it from an identity involving Pfaffians. Does he want to enumerate planar maps of some kind? He solves functional equations for formal power series. And it is not just a matter of one worker's inclinations. Look at the towering structure of general graphical enumeration theory! It is built of permutation groups and their cycle indices, and its pinnacles are formal power series [**6**].

So at times I gaze into the Future and contemplate a Mathematics in which there is no Graph Theory. That has been absorbed into Linear Algebra [**5**], or perhaps the Theory of Formal Power Series. But this mood does not last, since I am naturally optimistic. My vision usually ends with a glorious resurrection in the form of Matroid Theory.

I like matroids. I think of them as combinatorial objects of the same general kind as graphs, – generalizations of graphs in fact, – and even more desirable because they always have duals. It is true that I am not yet very good at drawing them, and if I thereby stand convicted of the feminine weakness of illogicality, then so be it. Matroid Theory brings with it out of the sea of Algebra "the abstract properties of linear dependence" and we discover paradoxically that fundamentally linear dependence is not an algebraic concept at all, even if it is at times decorated with fields and rings.

But I am getting ahead of my subject. Matroids are not discussed in Bondy and Murty. Still, we can always hope for a sequel, or an expanded Second Edition.

Meanwhile the present work gives us Graph Theory in its state of purity. It is really an outstanding book. Why, Appendix III alone (Some Interesting Graphs) is worth "a thousand pounds a puff".

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8. William Shakespeare, *As you like it*, Act II, Scene one.

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Asymptotic wave theory, by Maurice Roseau, Applied Mathematics and Mechanics, vol. 20, North Holland/American Elsevier, Amsterdam/New York, 1976, x + 349 pp.

The beauty of wave motion has long fascinated mankind even though he may not always have been aware that he was observing waves in action. The