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ON THE ISOMETRIES OF $L^p(\Omega, X)$

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The isometries of $L^p[0, 1]$, $1 \leqslant p < \infty$, $p \neq 2$, were determined by Banach [1, p. 178]. In that case every isometry T is of the form $(Tf)(\cdot) = u(\cdot)f(\phi(\cdot))$ where ϕ is a measurable transformation of $[0, 1]$ onto itself and u is a fixed function related by ϕ by the formula $|u|^p = d(\lambda \circ \phi)/d\lambda$ where λ is Lebesgue measure. Lamperti [4] determined the isometries of $L^p(\Omega)$ for any σ-finite measure space (Ω, Σ, μ). The result resembles Banach's except for the replacement of the point transformation ϕ by a set transformation. Cambern [3] determined the surjective isometries of $L^p(\Omega, K)$ for a separable Hilbert space K. These isometries resemble those of $L^p(\Omega)$ except for the emergence of an operator-valued function.

Our aim here is to describe the surjective isometries of $L^p(\Omega, X)$ for certain Banach spaces X (Theorem 1) and the *injective* isometries of $L^p(\Omega, K)$ for a separable Hilbert space K (Theorem 2).

Let (Ω, Σ, μ) be a σ-finite measure space. A *set homomorphism* Φ is a map of Σ into itself, defined modulo null sets, which preserves set differences and countable unions. If, in addition, $\mu(\Phi(\delta)) = 0$ if and only if $\mu(\delta) = 0$, then Φ is called a *set isomorphism.* It can be shown that Φ induces a transformation, also denoted by Φ, on the space of measurable functions defined on Ω with values in a separable Banach space X.

A Banach space X is called the *l^p-direct sum* of two Banach spaces X_1 and X_2 if X is isometrically isomorphic to $X_1 \oplus X_2$ with $\|x_1 \oplus x_2\|^p = \|x_1\|^p + \|x_2\|^p$.

THEOREM 1. *Let T be an operator on $L^p(\Omega, X)$, $1 \leqslant p < \infty$, $p \neq 2$, where X is a separable Banach space, and assume that X is not the l^p-direct sum of two nonzero Banach spaces (for the same p). Then T is a surjective isometry if and only if*

$$(1) \qquad (Tf)(\cdot) = S(\cdot)h(\cdot)(\Phi(f))(\cdot) \qquad f \in L^p(\Omega, X),$$

where Φ is a set isomorphism of the measure space onto itself, S is a strongly measurable map of Ω into $B(X)$ with $S(t)$ a surjective isometry of X for almost all $t \in \Omega$, and $h = (d\nu/d\mu)^{1/p}$ where $\nu(\cdot) = \mu(\Phi^{-1}(\cdot))$.

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REMARK. If (Ω, Σ, μ) admits a set isomorphism different from the identity, then the condition that X is not a l^p-direct sum is a necessary, as well as a sufficient, condition for the conclusion of Theorem 1 to hold. In particular, this is the case for separable measure spaces. But we do not know of any measure space which does not have this property except the trivial one, i.e. that which consists entirely of a single atom.

THEOREM 2. *Let T be an operator on $L^p(\Omega, K)$, $1 \leqslant p < \infty$, $p \neq 2$, where K is a separable Hilbert space. Then T is an isometry of $L^p(\Omega, K)$ into itself if and only if T is of the form* (1) *where Φ is a set isomorphism of the measure space into itself and $S(t)$ is an isometry of K into itself for almost all $t \in \Omega$.*

Our main tool is the following result, valid for all p, $1 \leqslant p < \infty$ and all separable Banach spaces X.

LEMMA 3. *Let T be a bounded operator on $L^p(\Omega, X)$ such that T maps functions of almost disjoint support into functions of almost disjoint support, then $(Tf)(\cdot) = A(\cdot)(\Phi(f))(\cdot)$ where Φ is a set homomorphism and A is a strongly measurable map of Ω into $B(X)$.*

This lemma, together with [3, Lemma 2] implies Theorem 2. Our proof of Theorem 1 depends on first characterizing the hermitian operators of $L^p(\Omega, X)$ (Theorem 4), and then using the fact that if T is a surjective isometry and H is hermitian, then THT^{-1} is hermitian. This idea was first used by Lumer [5] to characterize the isometries of certain spaces.

Recall that a bounded operator H on a Banach space is called *hermitian* if $\|\exp(itH)\| = 1$ for all real numbers t (see [2]).

THEOREM 4. *Let X be a separable Banach space and H a bounded operator on $L^p(\Omega, X)$, $1 \leqslant p < \infty$, $p \neq 2$. Then H is hermitian if and only if $(Hf)(\cdot) = A(\cdot)f(\cdot)$ for a hermitian-valued strongly measurable map A of Ω into $B(X)$.*

Details will appear elsewhere.

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