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Unitary representations and harmonic analysis, an introduction, by Mitsuo Sugiura, Kodansha Ltd., Tokyo; John Wiley & Sons, New York, London, Sydney, Toronto, 1975, xiii + 402 pp., $34.00.

There is the story of the cellist, playing but a single note, who explained to his one friend and many enemies that, while they sought the golden sound, he had found it. Just so, those mathematicians penetrating the mysteries of harmonic analysis may well say to all the rest that while others seek, we have found the center of mathematical elegance.

It is, of course, well known to astrologers, economists, and those of us who plant seeds during the dark of the moon that everything goes in cycles, yet it was left for that *paure orphelin* Jean Baptiste Joseph Fourier to establish the matter beyond any shadow of doubt. Unfortunately his discoveries were so disconcerting as to cast misgivings on their utility for prediction. Everything goes in cycles, but the manner of its going changes from interval to interval and moment to moment in unsatisfactory ways. Yet, since Fourier's time, harmonic analysis has intersected most all mathematical problems short of that of forecasting the future. One establishes this fact both empirically and intellectually. The empirical proof consists in compiling even a minibiography whose length will be exceeded only by the brilliance of some of the contributors. The intellectual argument is simpler. Quote the words of Dieudonné spoken during the American Academy Workshop on the Evolution of Modern Mathematics to the effect: "–it is strange to study the work of Harish-Chandra in the last 15 years on representations of semisimple Lie groups. He uses such a fantastic arsenal of techniques taken from all over mathematics. It is quite clear that the number of people who are able to understand this work is very small at present, because it taxes the intellectual capacity of a person."

In defiance of the gods, Sugiura has written a fine book for mere mortals. Unfortunately one senses that it represents a good idea whose time has passed–the tide has come and gone, more than once, before its arrival. This feeling has a number of parts: First, the functional analysis required for its easy comprehension has been replaced in many quarters by an amorphous subject governed largely by topological interests and considerations. Second, there have been a large number of books and papers written on representation theory during the last decade, even during the last few years. Some of these are very good. Third, and most unfortunate of all, the long marriage of mathematics and physics seems destined for a final separation. We discuss these points in sequence.

Turning to the first, much of the functional analysis used by Sugiura is of a classical nature, according to a definition of Singer, having been around for more than ten years. In his introductory chapter, the author makes a very natural transition from the complex representation theory of a finite group to that of a compact group by means of the spectral theorem for a bounded selfadjoint operator on a Hilbert space, by a brief use of commutants and von Neumann algebras, and by a final appeal to the direct integral of Hilbert spaces. This indicates the level of soft analysis that Sugiura requires in his determination of the complex irreducible representations of six Lie groups: the one-dimensional torus group $\mathbf{T} = \mathbf{R}/2\Pi\mathbf{Z}$, the special unitary group SU(2), the 3-dimensional rotation group SO(3), the n-dimensional vector group $\mathbf{R}^n$, the Euclidean motion group $M(2)$ of the plane, the special linear group SL(2, **R**) of real 2×2 matrices of determinant one. Besides the calculation of the complex irreducible representations of these, he extends or finds analogues of a number of theorems of classical Fourier analysis. These include: (i) (*Riemann-Lebesgue*) the Fourier transform $\mathcal{F}$ maps L^1 into L^∞; (ii) (*Schwartz*) the Fourier transform $\mathcal{F}$ is a topological isomorphism of the space $\mathcal{S}$ of all rapidly decreasing functions on the group manifold G onto itself whose inverse is the conjugate transform $\mathcal{F}^*$; (iii) (*Paley-Wiener*) the Fourier-Laplace transform $\mathcal{F}^c$ is a topological isomorphism of the space $\mathcal{D}(G)$ of complex valued C^∞-functions on G with compact support onto the space $\mathcal{H}$ of entire functions of exponential type on G; and (iv) (*Parseval-Plancherel*) the Fourier transform, from a suitable point of view, is a unitary transform between Hilbert spaces related to the group. Here the mathematician familiar with only classical Fourier series and integrals tends to flounder.

The Fourier transform $\mathcal{F}f$ of a function f with domain the semisimple Lie group G is a map $\mathcal{F}f\colon \hat{G} \to B(H)$ from the *unitary dual* $\hat{G}$ of equivalence classes of complex irreducible representations of G into the union $B(\hat{H})$ of the bounded operators on the spaces of the representations occurring in $\hat{G}$. A description of the general situation here, requiring the efforts of some of the most gifted mathematicians of our time, has, to some extent, expanded into a subject touching virtually all of mathematics. Needless to say, Sugiura limits himself to discussing the basic idea of Plancherel measures for his particular groups. For even 2×2 matrices over locally compact fields, things tend to get out of hand as one can see for himself by looking into some of the results of Gelfand and his school. Furthermore, in addition to various techniques from classical soft analysis, the author makes use of a generous bag of tricks from hard analysis. Consequently, the reviewer has the disquieting suspicion that a high percentage of *well-trained* graduate students in analysis will lack both the motivation for reading such a book as well as the techniques for so doing.

Concerning its competitors, we mention only the works of Serge Lang, V. S. Varadarajan, and Garth Warner, although many other sources are presently available. By limiting himself to the six groups mentioned above, Sugiura avoids defining Lie groups, differentiable manifolds, Dynkin diagrams and the extensive surrounding apparatus. Strangely enough, this stripping the subject of its mysteries buys but a modest additional clarity while eliminating an interplay between structure and analysis on which so much of the attractiveness of the theory depends. Nevertheless, it highlights a fallacy some of us have held, to wit, that algebra, topology, and differentiable manifolds create the problems of harmonic analysis on Lie groups. To the contrary, hard analysis and attention to detail make it difficult. Sugiura's omission of the Lie theory weakens the case for using his book as a primer by foreclosing the possibility of seeing various concepts and definitions in their most rudimentary forms. A comparison of his book with that of Serge Lang, treating only SL(2, **R**), belies an amusing criticism of Lang's monograph on group cohomology. In the present instance, the lighter and more conceptual touch of Lang contrasts pleasantly with the remorseless attention to detail of Sugiura. In yet another book dealing in much greater depth with the structure of Lie groups and Lie algebras than either of these two, Varadarajan supplies a long list of problems after each chapter, suggesting its use as a textbook. Yet one imagines that only an instructor of the power of Varadarajan could successfully use his book with graduate students of customary ability. Finally, the two volumes of Warner are intended as an encyclopedia of the representation theory of semisimple Lie groups. The prospective reader need not be put off by Warner's astonishing list of prerequisites. In charity to his prospective audience, Warner should have included the Halmos prescription that a potential reader need not be unduly dismayed upon finding he does not have the prerequisites for reading the prerequisites.

Although these offerings are far from easy, the reviewer believes that a highly motivated person, with better than average, but less than genius ability, can learn representation theory from them over a period of several years without benefit of a teacher. However, the time is now ripe for a professional writer of mathematics, as contrasted to a research mathematician, to rewrite this material in such a fashion that an interested mathematician or physicist can learn much of representation theory without making such a heavy commitment. Strangely enough, in a field where one almost never comes to the end of the definitions and symbols, none of these books has a really good index or symbol table for the nonexpert. In this respect, Warner is the best although in others his book is the least tightly organized.

Finally, to consider our last point, we observe that the gap between mathematics and the hard sciences as well as engineering continues to widen at an increasing rate. This appears particularly true of mathematics and physics where communications dwindle with the passing decades as each field withdraws from natural problems into their own brand of abstractions. Consequently, one forgets that the lower harmonic analysis constituted a flourishing trade among theoretical physicists at one time. Newton attempted to determine the velocity of sound in air by means of elastic waves in a one-dimensional model. Correcting Newton's thermodynamic, rather than mathematical, mistake Laplace obtained a good approximation to the experimental value of the velocity. These early studies of one-dimensional discrete lattices culminated in Euler's investigation of the continuous string, leading to his discovery, not understood at the time, that an arbitrary function could be represented by a series in sines and cosines. Fifty years later, Fourier settled the resulting controversy by his famous discovery and exploitation of the same fact. Subsequently employing the model of Newton in his researches on the dispersion of light, Cauchy arrived at incorrect results since Newton's mechanical system enjoys special properties not enjoyed by light. One also recalls that each of the famous British physicists, Lords Kelvin and Rayleigh, found harmonic analysis indispensable in his work. Furthermore, there can be little doubt that the harmonic point of view culminated eventually in the discovery of quantum mechanics by Born, De Broglie, Heisenberg, Schrödinger, and others.

While some of the foremost modern practitioners of representation theory assert their inspiration lies in physics, few of them face up to the fact that physics is an experimental science so that theories are of maximal use confronting numbers experimentalists observe in the laboratory. For a long while, mathematicians have restricted their interest in numbers to statements such as: there exist no nonvanishing vector fields on spheres of even dimension or that the set of isomorphism classes of k-dimensional vector bundles over a paracompact space B has a natural bijective correspondence with the set of homotopy classes of mappings of B into the Grassmann manifold of k-dimensional subspaces of an infinite dimensional space. We have passed the art of computation along to computerologists–selling both ourselves and the world out.

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Vector measures and control systems, by Igor Kluvanek and Greg Knowles, North-Holland Mathematics Studies, vol. 20, North-Holland/American Elsevier, Amsterdam/New York, 1975, ix + 180 pp., \$13.50.

The theory of vector measures has been under increasingly heavy study for the last decade. By the early seventies coherent bodies of knowledge had solidified in the areas of vector measure theory that grew from either the Orlicz-Pettis theorem or the Dunford-Pettis Radon-Nikodým theorem for the Bochner integral. But as late as 1974 the range of a vector measure was still an object of some mystery.

At that time the two main theorems about the range of a vector measure were Liapunov's convexity theorem (the range of a nonatomic vector measure with values in a finite dimensional space is compact and convex) and the Bartle-Dunford-Schwartz theorems (a vector measure with values in a Banach space has a relatively weakly compact range and is absolutely continuous with respect to a scalar measure). The infinite dimensional version of Liapunov's theorem remained a particular enigma; Liapunov had shown, by example, that his convexity theorem failed for vector measures with values in the sequence spaces l_p $(1 \leqslant p \leqslant \infty)$. The very scope of Liapunov's example served to block serious research into the infinite dimensional version of Liapunov's convexity theorem. This, in turn, held up the understanding of the bang-bang principle for control systems with infinitely many degrees of freedom (e.g. a control system governed by a partial differential equation).

Also in the early seventies it became clear that a sharpened form of the Bartle-Dunford-Schwartz theorem was needed. It was realized that the range