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MONOTONICITY AND UPPER SEMICONTINUITY

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Introduction. We show in this note that set valued maximal monotone operators on a Hilbert space possess the upper semicontinuity property called property (Q), introduced by Cesari [2] and used extensively in the existence analysis of optimal control theory. As a particular consequence we conclude rather easily, the known result (see [1], for example) that maximal monotone operators have closed graph and are thus demiclosed. As a simple application of this to optimal control theory we give an existence theorem for a Mayer problem. Details and extensions are found in [5] where we study upper semicontinuity in the context of semiclosure operators of general topology.

Notations. Let H be a Hilbert space with inner product $\langle \, , \rangle$ and induced norm $\|\cdot\|$. Let 2^H denote the collection of all nonempty subsets of H. As in [1], a set valued function $F: H \longrightarrow 2^H$ is said to be maximal monotone, if its graph $G(F)$ is maximal with the property that $\langle y_2 - y_1, x_2 - x_1 \rangle \geqslant 0$ for all $(x_1, y_1), (x_2, y_2) \in G(F)$. As in [2], $F: H \longrightarrow 2^H$ is said to have property (Q) if for each $x_0 \in H$,

$$F(x_0) = \bigcap_{\delta > 0} \operatorname{cl} \operatorname{co} \bigcup \{F(x), \|x - x_0\| < \delta\} \tag{1}$$

where cl co A denotes the (strong) closure of the convex hull of A. It is seen that if F is monotone then the right hand side of equation (1) is also monotone and hence we obtain

THEOREM 1. *If $F: H \longrightarrow 2^H$ is maximal monotone, then F has property (Q).*

REMARKS. 1. It is to be noted that maximality is important in the above theorem. For example, if $F(x) = \{[x]\}$, x real, where $[x]$ is the greatest integer $\leqslant x$, then F does not have property (Q) at $x = 0$. On the other hand, F is monotone but not maximal since $I + F$ is not surjective; indeed $3 \neq x + [x]$ for any real x.

2. $F(x)$ is closed and convex for each $x \in H$, if F has property (Q) and hence if F is maximal monotone.

3. Using Banach-Saks-Mazur theorem it is seen that if F has property (Q), $y_k \longrightarrow y_0$ weakly, $x_k \longrightarrow x_0$ strongly, and $y_k \in F(x_k)$, then $y_0 \in F(x_0)$. By

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Theorem 1, maximal monotone operators are also thus demiclosed.

4. The above statements can be modified to the case where the operators are not defined on the whole space H. In this case Theorem 1 would guarantee property (Q) only relative to the domain D of the operator. Also, the Hilbert space H may be replaced by a reflexive Banach space B, in which case the operators take values in 2^{B^*}, where B^* is the dual of B. Monotonicity is defined in terms of the "action" $\langle y, x\rangle = y(x)$ for $x \in B$ and $y \in B^*$.

5. The above theorem suggests carrying over the results of maximal monotone operators to set valued functions with property (Q). It is of interest to note, for example that $aI + bF$, a, b real, has property (Q) if F does; I being the identity map. However, we cannot generalize the important result of Minty that if F is maximal monotone then $I + F$ is surjective. Indeed, if $F(x) = \{-x\}$, real, then F has property (Q), but $(I + F)x = 0$ for all x and $I + F$ is not surjective.

AN APPLICATION (see [3]). Let $B_i, i = 1, 2, 3, 4$ be Banach spaces and let B_1 and B_4 be reflexive. For each $y \in B_2$, let $U(y) \subset B_3$, be given. Let $L\colon B_1 \longrightarrow B_4$ be such that $L(A)$ is bounded if A is bounded in B_1 and such that graph of L is closed in (weak, weak) topology. Let $M\colon B_1 \longrightarrow B_2$ be such that if $x_k \longrightarrow x_0$ weakly in B_1, then $Mx_k \longrightarrow Mx_0$ strongly in B_2. Let $F(y) = \{f(y, u)| u \in U(y)\}, y \in B_2$. For $d > 0$, let $\Sigma_d = \{(x, u)| x \in B_1, \|x\| \leqslant d, u \in U(Mx), Lx = f(Mx, u)\}$.

THEOREM 2. *With the above notation, let $B_4 \subset B_2^*$, dual of B_2. Let $F(y)$ be maximal monotone on B_2. Let $r\colon B_1 \longrightarrow R$, reals, be a given weakly lower semicontinuous functional with $\inf\{r(x), \|x\| \leqslant d\} > -\infty$ for each $d > 0$. Then, for any $d > 0$ with Σ_d nonempty, the functional $S(x, y) = r(x)$ attains its minimum on Σ_d.*

REMARKS. 1. If B_2 is a Hilbert space and ϕ is a lower semicontinuous proper convex function (see [1]) then the subdifferential $\partial\phi(y), y \in B_2$ is maximal monotone. If we take $f(y) = Ay + u, u \in U(y) = \partial\phi(y)$ then $F(y) = f(y, U(y))$ is maximal monotone if A is so and $\operatorname{dom}(A) \cap$ interior $(\operatorname{dom}\phi) \neq \varnothing$. The same holds for $f(y, u) = Ay + ku, k > 0$ and $f(y, u) = Ay + J_\lambda u, \lambda > 0$, $J_\lambda = (I + \lambda\partial\phi)^{-1}$.

2. If $B_2 = B_4 \subset H$, a Hilbert space, if $U(y) = U$, fixed for $y \in B_2$ and if there is a $u \in U$ such that $f(y, u) = 0$ for all $y \in B_2$ then the relation $z \in y + F(y)$ is solvable for every $z \in B_2$ and by Minty's theorem, F is maximal monotone if we know that it is monotone. This situation is seen in the following examples.

(i) H is a real separable Hilbert space and $\{\phi_i\}$ is a complete orthonormal system in H. Let $B_2 = \{\Sigma_{i=0}^\infty c_i\phi_{2i} | c_i \text{ real}, i = 1, 2, \ldots\}$. Then B_2 is a closed subspace of H and $B_2^\perp = \{\Sigma_i d_i\phi_{2i+1} | d_i \text{ real}, i = 1, 2, \ldots\}$. Let $U(y) = U = B_2$

for all y. Let $\lambda_i > 0$, $i = 1, 2, \ldots$ be a given sequence of reals. For $y \in B_2$ and $u \in U$, let $f(y, u) = \Sigma c_i d_i \lambda_i \phi_{2i+1}$ where $y = \Sigma c_i \phi_{2i}$ and $u = \Sigma d_i \phi_{2i+1}$. Clearly $f(y, u) \in B_2$ and hence $\langle f(y, u) - f(z, v), y - z\rangle = 0$ and thus $F(y)$ is monotone. Since $f(y, 0) = 0$ for all y it turns out by above remarks that F is maximal.

(ii) Let B_2 be the closed subspace of $L_2([0, 2\pi])$ generated by $\{\sin nt, n = 0, 1, 2, \ldots\}$. Let B_4 = closed subspace generated by $\{\cos nt, n = 0, 1, \ldots\}$. Let $U = U(y) = B_4$ for all y. Let $f(y, u) = u \cdot \int_0^t y(\tau)d\tau$. Then $f(y, u) \in B_4$ and $F(y) = f(y, U)$ is maximal monotone, as before.

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