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DISINTEGRATION OF MEASURES ON COMPACT TRANSFORMATION GROUPS

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The present work falls into two parts. In the first, a left transformation group [2] (G, X) with G a compact *metric* group and X a locally compact Hausdorff space is given; in the second, a bitransformation group [2] (G, X, T) with G, X compact Hausdorff and T arbitrary is considered. It is always assumed that G acts *freely*; thus $g \cdot x = x$ implies $g =$ identity in G $(x \in X)$.

1. Let $\pi: X \longrightarrow X/G \equiv Y$ be the projection. Let μ be a Radon measure on X, $\nu = \pi(\mu)$.

1.1. THEOREM. *There is a disintegration* [1], $\lambda: y \longrightarrow \lambda_y$ *of* μ *with respect to* π *such that*

(a) λ_y *is supported on* $\pi^{-1}(y)$;

(b) λ *is* ν*-Lusin-measurable*

(*thus, if* $K \subset Y$ *is compact, there is a countable collection* K_i *of compact sets, with* $\nu(K \sim \bigcup_{i=1}^{\infty} K_i) = 0$, *such that* $\lambda|K_i$ *is continuous for each* i). *If* λ' *is another disintegration of* μ *with respect to* π *satisfying* (a) *and* (b), *then* $\lambda' = \lambda$ ν*-a.e.*

To prove 1.1, one first assumes X is compact and G is a Lie group. In this case, X is "measure-theoretically" the product $Y \times G$; this follows from the existence of local cross-sections to the projection π [6]. Let $\pi_2: X \cong Y \times G \longrightarrow G$, and define a map ξ from $L^1(Y, \nu)$ to the space of Radon measures on G as follows: $\xi(f) = \pi_2[(f \circ \pi) \cdot \mu]$. Apply the Dunford-Pettis Theorem [3] to ξ to obtain a map ω from Y to $M_+(G) =$ the set of positive Radon measures η on G such that $\|\eta\| = 1$. The map λ is easily obtained from ω. One now completes the proof by (i) approximating G by a *sequence* of Lie groups [6]; (ii) using the fact that there is a locally countable collection of pairwise disjoint compact subsets of Y the complement of whose union is locally ν-null [1].

2. First suppose G is metric. Let μ be a *T-ergodic* measure on X, and let λ be a disintegration of μ as in 1.1. Let $G \supset G_0 = \{g \in G | \int_X f(gx)\, d\mu(x) = \int_X f(x)\, d\mu(x)$ for all $f \in C(X)\}$; G_0 is a closed subgroup of G. Denote the normalized Haar measure on G_0 by γ_0.

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2.1. THEOREM. *For each $y \in Y$, there exists $x \in \pi^{-1}(y)$ such that $\int_X f\, d\lambda_y = \int_G f(gx)\, d\gamma_0(g)$* $(f \in C(X))$.

Thus each λ_y "looks like" γ_0.

To prove 2.1, define ϕ_x: $G \longrightarrow X$: $g \longrightarrow g \cdot x$ for each $x \in X$. Then ϕ_x is a homeomorphism onto $\pi^{-1}\pi(x)$. Define F: $X \longrightarrow M_+(G)$: $F(x) = \phi_x^{-1}(\lambda_y)$ where $y = \pi(x)$.

2.2. LEMMA. *Let H map X to a Hausdorff space, and suppose* (i) *H is μ-Lusin-measurable,* (ii) *$H(xt) = H(x)$ μ-a.e. for each fixed $t \in T$. Then $H =$* const *μ-a.e.*

This lemma may be applied to F; thus $F(x) =$ const μ-a.e. Results of [5] now imply that $F(x) = \gamma_0$ μ-a.e., and 2.1 follows immediately.

If G is not metric, our results are quite a bit weaker. We do have an analogue of 2.1, however, if Y has a strong lifting [3]. Let $M_0(X) = \{\eta$: η is a positive Radon measure on X, $\|\eta\| = 1$, η is G_0-invariant$\}$. It is easily seen that 2.1 is equivalent with the following

2.3. STATEMENT. For each $y \in Y$, λ_y is extreme in the compact convex set $M_0(X)$.

2.4. THEOREM. *Suppose Y has a strong lifting* [3]. *Then there exists a weakly measurable disintegration λ such that* (i) *λ_y is supported on $\pi^{-1}(y)$*; (ii) *λ_y is extreme in $M_0(X)$ for all y.*

The proof is a straightforward argument using 2.1 and approximation of G by Lie groups.

These results represent a portion of the author's doctoral thesis [4]. He wishes to thank his advisor, Professor Robert Ellis, for his many valuable suggestions and his constant encouragement.

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