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AN AVERAGING PROPERTY OF THE RANGE OF A VECTOR MEASURE

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Our discussion centers around the striking properties displayed by the range of a vector-valued measure. Let Σ be a σ-field of sets, X be a Banach space and $F: \Sigma \longrightarrow X$ be a countably additive map (a vector measure). Bartle, Dunford and Schwartz [3] showed that $F(\Sigma)$ is relatively weakly compact; Liapounov [13] (see also Lindenstrauss [14]) showed that if X is finite dimensional then $F(\Sigma)$ is compact and, if F has no atoms, convex. Some additional peculiarities: Each extreme point of the closed convex hull of $F(\Sigma)$, $\overline{\text{co}}(F(\Sigma))$, lies in $F(\Sigma)$ [12]. Each extreme point of the closed convex hull of $F(\Sigma)$ is a denting point of $\overline{\text{co}}(F(\Sigma))$ [1]. The exposed points of $\overline{\text{co}}(F(\Sigma))$ are strongly exposed [1] and a point $x \in \overline{\text{co}}(F(\Sigma))$ is exposed by $x^* \in X^*$ (the dual of X) if and only if F is $|x^*F|$-continuous. While any two dimensional unit ball is the range of a vector measure, the unit ball of an l_p^3 $(1 \leq p < 2)$ is not ([4], [7]). Kluvanek [10] has noted that as a consequence of a classical theorem of Banach [8] the unit ball of l_2 is the range of a vector measure; he [11] has also obtained a characterization of the range of vector measures. The closed unit ball of L_p (or l_p) for $1 < p < 2$ is *not* the range of a vector measure. Since this last assertion seems not to be easily deducible from Kluvanek's characterization, a few remarks on its proof are in order: Note that if the ball of X is the range of a vector measure F then X is the quotient via integration of the Banach space $B(\Sigma)$ of bounded Σ measurable functions—a $C(K)$ space. If X is also a subspace of some L_1 space then Grothendieck's inequality [15] implies X is isomorphic to a Hilbert space. Since $L_p[0, 1]$ is isomorphic to a subspace of $L_1[0, 1]$ by [5] but is not isomorphic to any Hilbert space [2] our original assertion follows.

Our main result is built upon the beautiful paper of Szlenk [16] and using his methods we have the

THEOREM. *Every sequence in the range of a vector measure has a subsequence whose arithmetic means are norm convergent.*

OUTLINE OF PROOF. Key to the proof is the fact proved by Bartle, Dunford and Schwartz [3] that there exists a probability measure μ on Σ with the same null sets as F. Look at a sequence $(F(E_n))$ chosen from the range of the vector

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measure F; there exists a sequence (x_m^*) in X^* such that for each x in the closed linear span of $\{F(E_n)\}$, $\|x\| = \sup_m\{|x_m^*(x)|\}$, where $\|x_m^*\| = 1$ for all m. For each n, let $F_n: \Sigma \longrightarrow X$ be defined by $F_n(A) = F(E_n \cap A)$. Each F_n is an X-valued countably additive μ-continuous measure. Moreover, the family $\{x_m^* F_n: m, n = 1, 2, \ldots\}$ is uniformly absolutely continuous with respect to μ and so by the Radon-Nikodym Theorem this family can be viewed as a uniformly integrable bounded subset of $L_1(\mu)$. Given any increasing sequence (n_k) of positive integers, it is easily established that

$$\left\|\frac{1}{p}\sum_{k=1}^{p} F(E_{n_k}) - \frac{1}{q}\sum_{k=1}^{q} F(E_{n_k})\right\| \leqslant \sup_m \left\|\frac{1}{p}\sum_{k=1}^{p} x_m^* F_{n_k} - \frac{1}{q}\sum_{k=1}^{q} x_m^* F_{n_k}\right\|_{L_1(\mu)}.$$

From this it is easy to mimic the closing steps (Lemma 2 and Theorem) of Szlenk [16] to obtain the desired conclusion.

COROLLARY. *A weakly compact order interval in a Banach lattice is the range of a vector measure, consequently it has the Banach-Saks property.*

PROOF. If $\langle 0, x\rangle$ is an order interval, then the gauge $\| \ \|_x$ of $\langle -x, x\rangle$ is a lattice norm on the linear span L_x of $\langle -x, x\rangle$. The completion C of $(L_x, \| \ \|_x)$ is an M-space with unit and so is a $C(K)$-space by Kakutani's representation theorem [9]. If $\langle 0, x\rangle$ is weakly compact, the canonical inclusion of $(L_x, \| \ \|_x)$ into X extends to a weakly compact linear operator from C to X which takes the non-negative members of the unit ball of C onto $\langle 0, x\rangle$. It is a well-known fact that $\langle 0, x\rangle$ is the closed convex hull of the range of the representing vector measure for this extension of the inclusion map (Bartle, Dunford and Schwartz [3]). We now apply the observation of Kluvanek and Knowles [12, Chapter 5, §5] to complete the proof.

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