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INDECOMPOSABLE MODULES: AMALGAMATIONS

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The determination of criteria for amalgamations (pushouts) of indecomposable modules to be indecomposable is a well-known central problem of representation theory upon which not much progress has been made. Here we announce two results on amalgamations, and describe some of their applications to the representation theory of Artin rings. We refer the reader to [3] for the notions of modules with cores and modules with cocores, recalling that such modules are indecomposable. In the cited paper basic modules were also discussed—we mention that a module B of finite length is basic precisely when $B/\text{rad } B$ is simple. Our first result is valid in any module category.

THEOREM 1. *Let M_1 be indecomposable and θ_i: $A \longrightarrow M_i$ proper monomorphisms, $i = 1, 2$, such that* $\text{Hom}(M_1, M_2/\text{im } \theta_2) = 0$. *Let X be the pushout of θ_1 and θ_2.*

(i) *If M_2 is indecomposable and* $\text{Hom}(M_2, M_1/\text{im } \theta_1) = 0$, *then X is indecomposable.*

(ii) *If $M_2/\text{im } \theta_2$ is indecomposable then X is indecomposable if and only if there is no homomorphism f: $M_2 \longrightarrow M_1$ such that $f\theta_2 = \theta_1$.*

The only result we know of bearing any resemblance to our second result is a lemma of Ringel [5, p. 313].

THEOREM 2. *If*

$$0 \longrightarrow S \longrightarrow B_1 \oplus B_2 \xrightarrow{(\rho_1, \rho_2)} M \longrightarrow 0$$

is a nonsplit exact sequence where S is a simple module and B_1 and B_2 are basic modules of finite length, then M is indecomposable if and only if neither ρ_1 nor ρ_2 is a split monomorphism.

If S is a simple submodule of a nonsimple basic module B over a homomorphic image of a hereditary left Artin ring such that $\text{Ext}^1(B, B) = 0$ then, using Theorem 2, it is readily checked that the cokernel of some homomorphism $S \longrightarrow B \oplus B$ is indecomposable if and only if $\text{Ext}^1(B/S, B) \neq 0$. When $\text{Ext}^1(B, B) \neq 0$ we have

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THEOREM 3. *If B is a basic module over a homomorphic image of a hereditary left Artin ring then every nonsplit extension of B by B is indecomposable.*

We conjecture that the ring has infinite representation type whenever $\mathrm{Ext}^1(B, B)$ has dimension > 1 over the division ring $(\mathrm{End}\, B)^{\mathrm{op}}$. We also conjecture that the ring specified in the following theorem has infinite representation type.

THEOREM 4. *Let R be a left Artin ring such that some nonbasic R-module has a core with nonsimple socle. Then there exists a finitely generated indecomposable R-module that does not have a cocore.*

The conjecture that R has infinite type is valid when R is an Artin algebra with square zero radical. The proof of Theorem 4, just as the proof of the existence theorem announced in [3], is via a construction based on Theorem 2. The next result, which is also known for radical squared zero Artin algebras [1], is an easy consequence.

THEOREM 5. *Every nonsimple indecomposable module over a left Artin ring has a waist if and only if every indecomposable module is either basic or has a simple socle.*

Our final results are useful in the classification of radical squared zero Artin algebras such that all indecomposable modules have cores or cocores—see [3]. The first of these is a consequence of Theorem 1. It enables one to construct with ease, albeit in a restricted context, new indecomposable modules from known ones.

THEOREM 6. *Let $M_1, \ldots, M_{n+1}$ be indecomposable modules over a left Artin hereditary ring of Loewy length 2 such that* $\mathrm{Hom}(M_i/\mathrm{rad}\, M_i, M_j/\mathrm{rad}\, M_j) = 0$ *for $i < j$. Let $A_1, \ldots, A_n$ be nonzero modules that admit superfluous monomorphisms $u_i\colon A_i \longrightarrow M_i$ and $v_i\colon A_i \longrightarrow M_{i+1}$. Then* coker w *is indecomposable where $w\colon \coprod_1^n A_i \longrightarrow \coprod_1^{n+1} M_i$ is the matrix*

$$\begin{pmatrix} u_1 & 0 & & & \\ v_1 & u_2 & & 0 & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & 0 & & \cdot & \cdot \\ & & & v_{n-1} & u_n \\ & & & & v_n \end{pmatrix}$$

We should point out that coker w can be obtained as an iterated amalgamation from the pushout diagrams

$$\begin{array}{ccc} A_i & \xrightarrow{v_i} & M_{i+1} \\ \downarrow \beta_{i-1}u_i & & \downarrow \beta_i \\ X_i & \longrightarrow & X_{i+1} \end{array} \qquad (X_1 = M_1, \beta_0 = 1_{M_1});$$

that is, coker $w \cong X_{n+1}$.

Now let S be a simple module over a radical squared zero Artin algebra R, let V be a submodule of the injective envelope of S properly containing S, and let $\coprod_1^{n+1} f_i: \coprod_1^{n+1} P_i \longrightarrow V$ be a projective cover of V where the P_i are indecomposable projective modules. Then there exist maps $\epsilon_i: S \longrightarrow P_i$ such that $g_i\epsilon_i = 1_S$, where $g_i = f_i|_{\mathrm{im}\,\epsilon_i}$. Denoting the cokernel of

$$\begin{pmatrix} \epsilon_1 & 0 & & & \\ -\epsilon_2 & \epsilon_2 & & & \\ 0 & \cdot & \cdot & & \\ & & \cdot \quad \cdot & 0 & \\ & & \cdot & \cdot & \\ & 0 & & -\epsilon_n & \epsilon_n \\ & & & & -\epsilon_{n+1} \end{pmatrix} : \coprod_1^n S \longrightarrow \coprod_1^{n+1} P_i$$

by $M(V)$ and the core of a module Y by $C(Y)$ we have

THEOREM 7. *If R is a radical squared zero Artin algebra of finite representation type, then*

(a) $C(M(V)) \cong \begin{cases} P_1 & \textit{if } V \textit{ is a basic module } (n = 0), \\ S & \textit{if } V \textit{ is not a basic module}; \end{cases}$

(b) *if X is a nonsimple R-module with a core then there is a unique V admitting a surjection ω: $M(V) \longrightarrow X$ such that* $\ker \omega \subseteq \mathrm{rad}\, M(V)$ *and* $\omega C(M(V)) = C(X)$.

We remark that results of Dlab and Ringel [2] and Müller [4] show that the construction of $M(V)$ depends only on V. We also remark that Theorem 7 classifies modules with cores over radical squared zero Artin algebras of finite type. From its proof, which utilizes another theorem characterizing exactly when certain mildly restricted amalgamations have cores, we get

COROLLARY 8. *If R is a radical squared zero Artin algebra of infinite representation type such that $V/\mathrm{rad}\, V$ is a direct sum of nonisomorphic simple modules, then $M(V)$ has a core.*

Proofs of results announced here will appear in a monograph on modules with cores.

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