instance the Lévy-Khintchin representation formula for negative definite functions is proved only in the symmetrical case (following Harzallah's method). There are other examples. And it is still a challenge for an analyst to discover a nonprobabilistic proof of the most difficult part of the Port and Stone theorem in the unsymmetrical case (the real case is much simpler and had been solved before by Beurling and myself).

The book is clear. Each of the 18 paragraphs starts with a short outline and finishes with a sufficient, but not exhaustive, bibliography. It is written with great care and there are very few misprints. Its reading is easy and enjoyable. Some straightforward proofs are left as exercises to the reader, even when the corresponding results are subsequently used. Several simple and illuminating examples are thoroughly detailed. To sum up: a highly recommendable introduction to the general potential theory from an "analytic" point of view.

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Stochastic differential equations and applications, Vol. 1, by Avner Friedman, Academic Press, New York, San Francisco, London, 1975, xiii + 231 pp., \$24.50. Vol. 2, by Avner Friedman, Academic Press, New York, San Francisco, London, 1976, xiii + 299 pp., \$32.50.

A diffusion process with values in R^d defined for some interval $[0, T]$ of time is a Markov process with R^d for its state space which has almost surely continuous trajectories. The conditional distribution of an infinitesimal increment $x(t+h) - x(t)$ of such a process given the past history $\{x(s)\}$ for $0 \leqslant s \leqslant t$ is supposed to be approximately Gaussian with mean $hb(t, x(t))$ and covariance $ha(t, x(t))$. For each t and x, $b(t, x)$ is a vector with components $\{b_j(t, x)\}$ and $a(t, x)$ is a symmetric positive semidefinite matrix with entries $\{a_{ij}(t, x)\}$. Although such a description may not hold for every diffusion process, one can describe a wide class of such processes in terms of their associated coefficients $\{a_{ij}(t, x)\}$ and $\{b_j(t, x)\}$. These are often referred to as diffusion and drift coefficients.

The problem then is to start with some given $\{a_{ij}(t, x)\}$ and $\{b_j(t, x)\}$ and then show that under suitable regularity conditions on a and b there corresponds to it a unique diffusion process. Since a Markov process is fully determined by its transition probabilities it is enough to construct the transition probability function $p(s, x, t, dy)$ from the coefficients a and b. One way of doing this is to look at some associated partial differential equations known as Kolmogorov's backward equations.

Let us fix a t in $0 < t \leqslant T$. For some fixed function $f(y)$ on R^d one considers the function $u(s, x)$ defined by

$$u(s, x) = \int f(y) p(s, x, t, dy) \quad \text{for } 0 \leqslant s \leqslant t. \tag{1}$$

Assuming that the function $u(s, x)$ is smooth, one shows that it satisfies the differential equation

$$(2)\qquad \frac{\partial u}{\partial s} + \frac{1}{2}\sum a_{ij}(s,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_j(s,x)\frac{\partial u}{\partial x_j} = 0$$

with the boundary condition

$$(3)\qquad \lim_{s\uparrow t} u(s,x) = f(x).$$

If the differential equation (2) can be solved with the boundary condition (3) for a sufficiently large class of functions $f(\cdot)$, then one can show that the solution $u(s,x)$ does have a representation (1) which yields the transition probability function $p(s,x,t,dy)$. One can then go on to show that it has the properties expected of it so that it can serve as the transition probability function of a diffusion process. Such a method, of course, reduces the problem of constructing a diffusion process for specified coefficients to one of solving the differential equation (2). There are some slight variations possible in that– one can consider related differential equations instead of (2). In order to study how the properties of $[a,b]$ affect the behaviour of the process one has to use the differential equation (2), determine properties of $u(s,x)$ and then translate them into properties of the process.

A direct method of constructing the process was introduced by Itô. Let us take a $d \times d$ matrix $\sigma(t,x)$ such that $\sigma(t,x)\sigma^*(t,x) = a(t,x)$ [* here denotes the transpose]. One way of generating a Gaussian random vector with mean hb and covariance ha is to take a Gaussian random vector $\xi(h)$ with mean 0 and covariance hI and then consider $\sigma\xi(h) + hb$. For $\xi(h)$ one can take an increment of a d-dimensional Brownian motion and let $\xi(h) = \beta(t+h) - \beta(t)$. One can then try to solve

$$x(t+h) - x(t) \sim \sigma(t,x(t))(\beta(t+h) - \beta(t)) + hb(t,x(t)).$$

This can be abbreviated as

$$(4)\qquad dx(t) = \sigma(t,x(t))\,d\beta(t) + b(t,x(t))\,dt.$$

Actually one looks at the associated stochastic integral equation

$$(5)\qquad x(t) = x_0 + \int_{t_0}^{t} \sigma(s,x(s))\,d\beta(s) + \int_{t_0}^{t} b(s,x(s))\,ds$$

for $t_0 \leqslant t \leqslant T$.

There are problems defining these integrals because $\beta(t)$ is not of bounded variation in t. Itô developed the theory of stochastic integrals to give meaning to the integrals of the type that occur in (5). He then proved that the stochastic integral equation (5) has a unique solution provided σ and b are Lipschitz continuous in x and grow at most linearly in x [uniformly in t]. He also showed that for each fixed x_0 and t_0 the solution $X_{x_0,t_0}(t)$ of (5) is a Markov process with continuous trajectories. Moreover the transition probability function of the process, $p(s,x,t,A)$ is given by

$$p(s,x,t,A) = \text{Prob}\,[X_{s,x}(t) \in A].$$

Here Prob refers to Brownian motion probability and one views $X_{s,x}(t)$ as a functional of $\beta(\cdot)$. Now one has a purely probabilistic method of constructing the process from the coefficients. The connection with partial differential equations is still present in the form of Itô's formula which prescribes the chain rule for stochastic differentials. This formula states that if $u(t,x)$ is smooth and $x(t)$ satisfies (4) then

$$du(t,x(t)) = g(t,x(t))\,dt + G(t,x(t)) \cdot d\beta(t) \tag{6}$$

where

$$g(t,x) = \frac{\partial u}{\partial t} + \frac{1}{2}\sum a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_j(t,x)\frac{\partial u}{\partial x_j} \tag{7}$$

and

$$G(t,x) = (\nabla u \cdot \sigma)(t,x). \tag{8}$$

The term with the second derivatives on the right side of (7) is the additional term not found in the usual chain rule of ordinary calculus. The relation (6) is, of course, established rigorously in its integrated version.

One consequence of (6) is that

$$u(t,x(t)) - \int_{t_0}^{t} g(s,x(s))\,ds \tag{9}$$

is a martingale. If $x(t)$ is the solution starting from x_0 at time t_0 then

$$u(t_0,x_0) = E\left[u(t,x(t)) - \int_{t_0}^{t} g(s,x(s))\,ds\right]. \tag{10}$$

If g is identically zero this provides the connection with Kolmogorov's backward equation. We have here an advantage in that we can use stopping times in the martingale (9). This enables one to solve the backward equation in suitable regions with boundary conditions.

One can now study the properties of the diffusion process directly by the equation (5). They have consequences for the solutions of the differential equation (2). One can use suitable test functions $u(s,x)$ and use (10) to deduce properties of the process. In fact one can go back and forth and use techniques from partial differential equations and stochastic differential equations to obtain properties of solutions of either.

This is the central theme of these two volumes where the relation is systematically used to establish results concerning solutions of partial differential equations and stochastic differential equations. We will now describe the contents of the two volumes.

Volume I has Chapters 1 through 9 and Volume II, Chapters 10 through 17. Chapters 1–5 are standard material. Some of the topics covered are the following: basic notions in stochastic processes, Markov processes and martingales, Brownian motion and its properties, stochastic integrals, existence and uniqueness of solutions of stochastic differential equations, Itô's method for constructing diffusions corresponding to given coefficients, Itô's formula.

Chapters 6, 10 and 15 deal mostly with partial differential equations. Chapters 6 and 10 deal with the elliptic case, i.e. when a is positive definite. Chapter 15 deals with the general case. Properties of solutions of the backward and other related equations, and the transition probability $p(s, x, t, dy)$ are established. Some times another text is referred to for the details of the proof. These results are used repeatedly in the book. In Chapter 6 a representation similar to (10) of the solution of the differential equation generalizing the backward equation is derived. In Chapter 7 the mutual absolute continuity of processes corresponding to the same diffusion coefficient, but different drifts, is established and the Radon-Nikodym derivative is computed.

Chapter 8 studies the behaviour of the solution of (5) for large t [one takes $T = \infty$]. The asymptotic behaviour of $E\|x(t)\|^2$ for large t is investigated in terms of the asymptotic behaviour of the coefficients in t and x.

The remaining chapters, except the last two, deal exclusively with the homogeneous case, i.e. when the coefficients do not depend explicitly on the time variable s. The transition probability $p(s, x, t, dy)$ is, in this case, of the form $p(t - s, x, dy)$ with a possible density (under additional assumptions) $p(t, x, y)$. Chapter 9 deals with the question of the recurrence or transience of the process in terms of the behaviour of the coefficients for large x. The methods are precise enough to distinguish between dimensions. Chapter 11 studies the question of when a diffusion in R^d will avoid with probability one sets of lower dimensions. The common method in Chapters 8, 9 and 11 is to construct suitable test functions and use relation (9) or its analog.

Chapters 12 and 13 treat the case when the diffusion coefficients could conceivably degenerate near the boundary of a region G. Then the process starting in G could either reach the boundary in a finite time, or approach a definite point on the boundary as $t \to \infty$ or spiral towards the boundary. The precise behaviour is studied in Chapter 12 and then is used in Chapter 13 to study Dirichlet type problems involving the basic operator

$$L = \frac{1}{2} \sum a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum b_j(x) \frac{\partial}{\partial x_j}.$$

The results are more detailed when $d = 2$.

Chapter 14 is concerned with the behaviour as $\varepsilon \to 0$ of the process corresponding to $[\varepsilon a, b]$. The process approaches the deterministic process corresponding to $[0, b]$. Probabilities of sets of trajectories away from the limiting deterministic one will go to zero. The precise exponential rate of decay as $\varepsilon \to 0$ is computed. This is used to study the behaviour of the transition probability density $p_\varepsilon(t, x, y)$ of the process and the behaviour of solutions of the Dirichlet problem

$$\frac{\varepsilon}{2} \sum a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_j(x) \frac{\partial u}{\partial x_j} = 0 \quad \text{in } G,$$

$$u(x) = f(x) \quad \text{on } \delta G$$

as $\varepsilon \to 0$. Of course the main interest is when the drift term is pointing inward at the boundary of G. These methods are also used to study the asymptotic behaviour of the first eigenvalue of the operator in the above equation in the region G as $\varepsilon \to 0$.

Chapters 16 and 17 again deal with the inhomogeneous case. They are concerned with some optimization and minimax problems. In Chapter 16 a suitable functional of stopping times is to be optimized over all stopping times. A description of the optimal stopping time is given in terms of solutions of certain variational inequalities. When it is a minimax type problem involving two stopping times, existence of a saddle point is proved and a description of it is given. In Chapter 17 the drift coefficients $\{b_j(t, x)\}$ depend on parameters which can be controlled by different players. The question of existence of a saddle point is studied for the resulting stochastic differential game.

The methods use the theory of quasilinear parabolic differential equations.

The volumes cover a wide variety of results proved under varying assumptions. It may sometimes be hard for a nonexpert to get a feeling for the conditions in terms of their role in establishing the results. There are some misprints, on occasion even in the statement of the theorems.

These volumes contain a considerable amount of detailed information that is quite important in the study of diffusion processes. The methods should prove useful in studying other problems as well. There are a lot of exercises to make the reader familiar with the ideas developed in the text. Complete references are given to other related works, where some of the material can be found. Most of the results have been obtained by the author himself in recent years.

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Sieve methods, by H. H. Halberstam and H.-E. Richert, Academic Press, London, 1974, xiii + 364 pp., $26.00.

The modern theory of sieve method has developed gradually, with fits and starts, over the past sixty years. From the outset the literature was hard to read because of the complicated nature of the arguments, while in recent times many of the most important results have remained unpublished, making it almost impossible to be well informed. Moreover, the literature has become tangled, and fragmented by a lack of unifying perspective. Expository accounts of the subject have usually been restricted to specific aspects, and in many cases even these have made difficult reading. An exception to this is found in Halberstam and Roth [**6**, Chapter 4], where the general nature and